A NEW CHARACTERIZATION OF THE FIBONACCI NUMBERS

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1. INTRODUCTION

A theorem due to E. Zeckendorf (see [1] for proof and discussion) asserts that every positive integer n can be represented uniquely as a sum of distinct Fibonacci numbers such that no two consecutive Fibonacci numbers appear in the representation. With the definition $u_1 = 1$, $u_2 = 2$, and $u_{n+2} = u_{n+1} + u_n$ for $n \geq 1$, Zeckendorf's theorem yields an expansion for each positive integer n in the form

$$n = \sum_{1}^{\infty} \alpha_i u_i ,$$

where α_i is either 0 or 1 for each $i \geq 1$ and $\alpha_i \alpha_{i+1} = 0$ for $i \geq 1$. Thus each positive integer n can be associated with a binary sequence $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_i, \ldots$, where for given n, we see that $\alpha_i(n) = 1$ if u_i appears in the Zeckendorf expansion of n; otherwise $\alpha_i(n) = 0$. The constraint $\alpha_i \alpha_{i+1} = 0$ for $i \geq 1$ effectively states that two consecutive 1's can never appear in the binary sequence corresponding to n. For example, if $n = 17$, then $17 = 1 + 3 + 13 = u_1 + u_3 + u_6$, so that 17 is associated with the binary sequence 101001. (It is clear that each such expansion must have all zeros to the right of some $i = i_o$ depending on n and these noncontributing zeros are suppressed.)

The question arises as to what occurs if, instead of disallowing two consecutive non-zero coefficients in a Fibonacci expansion, we disallow two consecutive zero coefficients. In other words, we wish to consider representing an arbitrary positive integer n as a sum of distinct Fibonacci numbers,

$$n = \sum_{1}^{N} \beta_i u_i$$

with binary coefficients satisfying $\beta_N = 1$ and $\beta_i + \beta_{i+1} \geq 1$ for $i = 1, 2, \ldots, N-2$. In the following, Theorem 1 affords a result dual to the Zeckendorf theorem by showing that such an expansion always exists for every positive integer and moreover the expansion is unique under the imposed coefficient constraints. Theorem 2, which is our main result, shows that the expansion property of Theorem 1 together with the uniqueness requirement is sufficient to characterize the Fibonacci numbers. This converse theorem for the dual representation corresponds to Daykin's result [2] on the converse to Zeckendorf's theorem.

2. A DUAL-ZECKENDORF THEOREM

Definition 1: The Fibonacci sequence $\{u_i\}$ is defined by $u_1 = 1$, $u_2 = 2$, and $u_{n+2} = u_{n+1} + u_n$ for $n \geq 1$.

Lemma 1:

(a)
$$\begin{cases} u_{k+1} - 1 = u_k + u_{k-2} + u_{k-4} + \cdots + u_3 + u_1 & \text{for } k \text{ odd.} \\ u_{k+1} - 1 = u_k + u_{k-2} + u_{k-4} + \cdots + u_4 + u_2 & \text{for } k \text{ even.} \end{cases}$$

(b)
$$u_{k+1} = u_k + u_{k-2} + u_{k-4} + \cdots + u_{k-2p+2} + u_{k-2p+1} ,$$

where

$$p = 1, 2, \ldots, \frac{k}{2} \text{ for } k \text{ even and } p = 1, 2, \ldots, \frac{k-1}{2} \text{ for } k \text{ odd.}$$

(c)
$$u_{k+1} - 2 = \sum_{1}^{k-1} u_i \quad \text{for } k \geq 1, \quad \text{where } \sum_{m}^{\text{"}n\text{"}}$$

is defined to be zero for $n < m$.

Proof. The straightforward inductive proof is left to the reader.

Our first theorem, as noted in the introduction, is essentially a dual of the Zeckendorf theorem [1] :

Theorem 1: Every positive integer n has one and only one representation in the form

(1)
$$n = \sum_{1}^{k} \beta_i u_i ,$$

where the β_i are binary digits satisfying

(2) $$\beta_i \beta_{i+1} \geq 1 \quad \text{for } i = 1, 2, \ldots, k-2$$

and

(3) $$\beta_k = 1 \quad .$$

For a given positive integer n, the value of k is determined as the unique integer for which the inequality,

(4) $$u_{k+1} - 2 < n \leq u_{k+2} - 2 \quad ,$$

is satisfied.

Convention: It will be assumed without explicit mention in the remainder of the paper that all expansion coefficients (subscripted variables α and β) are binary digits, that is, digits having either the value zero or the value unity.

Proof of Theorem: Let n be a positive integer satisfying inequality (4). From (c) of Lemma 1, we obtain the equivalent inequality,

(5) $$\sum_1^{k-1} u_i < n \leq \sum_1^{k} u_i \quad ,$$

so that by the Zeckendorf theorem, the non-negative integer

$$\sum_1^{k} u_i - n$$

possesses an expansion in the form,

(6) $$\sum_1^{k} u_i - n = \sum_1^{\infty} \alpha_i u_i \quad ,$$

with coefficients satisfying $\alpha_i \alpha_{i+1} = 0$ for $i \geq 1$.

Note from (5) that

$$\sum_1^{k} u_i - n < \sum_1^{k} u_i - \sum_1^{k-1} u_i = u_k \quad ,$$

which implies $\alpha_i = 0$ for $i \geq k$ in (6). In particular, $\alpha_k = 0$. Hence, (6) may be rewritten

$$\sum_{1}^{k} u_i - n = \sum_{1}^{k} \alpha_i u_i \quad \text{with } \alpha_k = 0 ,$$

or

$$(7) \qquad n = \sum_{1}^{k} (1 - \alpha_i) u_i = \sum_{1}^{k} \beta_i u_i \quad \text{with } \beta_k = 1 ,$$

where we have defined $\beta_i = 1 - \alpha_i$ for $i = 1, 2, \ldots, k$. It is clear from the relations $\alpha_i \alpha_{i+1} = 0$ and $\alpha_i \alpha_{i+1} = (1 - \beta_i)(1 - \beta_{i+1})$ that $\beta_i + \beta_{i+1} \geq 1$ for $i = 1, 2, \ldots, k - 2$, as required.

To show uniqueness, assume there exists a positive integer n with two representations:

$$(8) \qquad n = \sum_{1}^{m} \beta_i u_i = \sum_{1}^{p} \beta_i' u_i ,$$

where $\beta_m = \beta'_p = 1$, $\beta_i + \beta_{i+1} \geq 1$ for $i = 1, 2, \ldots, m - 2$ and $\beta_i' + \beta_{i+1}' \geq 1$ for $i = 1, 2, \ldots, p - 2$.

If $m \neq p$, then we assume $m > p$ without loss of generality, and from the coefficient constraints and Lemma 1, we have

$$\sum_{1}^{m} \beta_i u_i \geq u_m + u_{m-2} + u_{m-4} + \cdots + u_{1,2} = u_{m+1} - 1 ,$$

while

$$\sum_{1}^{p} \beta_i' u_i \leq \sum_{1}^{p} u_i \leq \sum_{1}^{m-1} u_i = u_{m+1} - 2 .$$

(Here and in what follows, the subscript notation $u_{1,2}$ serves to indicate the final term in a sum, the value of the final term being either u_1 or u_2 depending on the parity of the index associated with the initial term in the sum.) This is in evident contradiction to (8) and we conclude $m = p$.

Now, define $\alpha_i = 1 - \beta_i$ and $\alpha_i' = 1 - \beta_i'$ for $i = 1, 2, \ldots, p$. Then $\alpha_i \alpha_{i+1} = \alpha_i' \alpha_{i+1}' = 0$ for $i = 1, 2, \ldots, p-1$, and (8) becomes

$$\sum_1^p (1 - \alpha_i)u_i = \sum_1^p (1 - \alpha_i')u_i$$

or

$$\sum_1^p \alpha_i u_i = \sum_1^p \alpha_i' u_i \quad . \tag{9}$$

Since both sides of (9) are admissible Zeckendorf representations, the uniqueness of such representations implies $\alpha_i = \alpha_i'$ for $i = 1, 2, \ldots, p$ or equivalently $\beta_i = \beta_i'$ for $i = 1, 2, \ldots, p$, which proves uniqueness of the dual representation and completes the proof of Theorem 1.

3. THE CONVERSE THEOREM

Next, we will show that the expansion property expressed in Theorem 1 actually provides a characterization of the Fibonacci numbers.

Definition 2: An arbitrary sequence of positive integers, $\{v_i\}$, $i = 1, 2, \ldots$ will be said to possess the dual unique representation property (Property D) if and only if every positive integer n has a unique representation in the form

$$n = \sum_1^p \beta_i v_i \quad \text{with} \quad \beta_p = 1, \quad \text{and} \tag{10}$$

$$\beta_i + \beta_{i+1} \geq 1 \quad \text{for } i = 1, 2, \ldots, p-2 \quad . \tag{11}$$

Corollary 1: A sequence $\{v_i\}$ possessing Property D has distinct elements; that is, $v_m \neq v_n$ for $m \neq n$.

Proof: Assume $m \neq n$ and $v_m = v_n$. Take $m > n \geq 1$ without loss of generality; then,

$$\sum_{\substack{i=1 \\ i \neq n}}^{m} v_i = \sum_{1}^{m-1} v_i ,$$

contradicting the assumed uniqueness of expansions satisfying (10) and (11).

Lemma 2: If $\{v_i\}$ possesses Property D, then $v_1 = u_1$ and $v_2 = u_2$, where $\{u_i\}$ is the Fibonacci sequence of Definition 1.

Proof: In order to represent the integer 1 in the proper form [(10)-(11)], it is clear that either v_1 or v_2 must be equal to 1. If $v_1 = 1$, then $v_2 = 2$ necessarily and the Lemma is proved. In the remaining case, $v_1 = 2$, $v_2 = 1$ and it follows that $v_3 = 3$ and $v_4 = 6$. At this point, it is impossible to represent the integer 8 in proper form no matter how the remaining (distinct) v_i are chosen. Thus $v_1 = 1 = u_1$ and $v_2 = 2 = u_2$ as stated.

Theorem 2. If $\{v_i\}$, $i = 1, 2, \ldots$ is an arbitrary sequence of positive integers possessing Property D, then $v_i = u_i$ for all $i \geq 1$.

Proof: The assertion is true for $i = 1$ and $i = 2$ as proved in Lemma 2. Now, assume as an induction hypothesis that $v_1 = u_1$, $v_2 = u_2$, ..., $v_k = u_k$ for some $k \geq 2$. We wish to show that $v_{k+1} = u_{k+1}$ necessarily.

Recall from Theorem 1 (noting $v_i = u_i$ for $i = 1, 2, \ldots, k$ by the inductive assumption) that every positive integer n satisfying

$$0 \leq n \leq \sum_{1}^{k} v_i$$

has a representation

$$n = \sum_{1}^{m} \beta_i v_i ,$$

where $m \le k$, $\beta_m = 1$ and $\beta_i + \beta_{i+1} \ge 1$ for $i = 1, 2, \ldots, m-2$.

We show first that $v_{k+1} \ge u_{k+1}$. For, if not, then $v_{k+1} < u_{k+1}$ and

(12) $v_{k+1} + v_{k-1} + v_{k-3} + \cdots + v_{1,2} < u_{k+1} + u_{k-1} + u_{k-3} + \cdots + u_{1,2} = u_{k+2} - 1$,

which implies

$$v_{k+1} + v_{k-1} + \cdots + v_{1,2} \le u_{k+2} - 2 = \sum_1^k v_i .$$

From (12) and the remark in the preceding paragraph, we have

$$v_{k+1} + v_{k-1} + v_{k-3} + \cdots + v_{1,2} = \sum_1^m \beta_i v_i ,$$

with $m \le k$, $\beta_m = 1$ and $\beta_i + \beta_{i+1} \ge 1$ for $i = 1, 2, \ldots, m-2$. Since both sides are in the proper form and are not identical, uniqueness is contradicted. Therefore $v_{k+1} \ge u_{k+1}$ as asserted.

Now assume $v_{k+1} > u_{k+1}$. We shall show that this assumption also leads to a contradiction of uniqueness. If $v_{k+1} > u_{k+1}$, then the unique representation of the integer

$$\sum_1^k u_i + 1$$

has the form

(13) $\displaystyle\sum_1^k u_i + 1 = \sum_1^{k+m} \beta_i v_i$ with $m \ge 2$, $\beta_{k+m} = 1$ and $\beta_i + \beta_{i+1} \ge 1$ for $i = 1, 2, \ldots, k+m-2$.

(For, if $m < 2$ in (13), then $m = 1$ since v_{k+1} must certainly appear with non-vanishing coefficient. But

$$\sum_1^{k+1} \beta_i v_i \ge v_{k+1} + u_{k-1} + u_{k-3} + \cdots + u_{1,2} > u_{k+1} + u_{k-1} + u_{k-3} + \cdots + u_{1,2} =$$

$$= \sum_1^k u_i + 1,$$ so that (13) could not be valid.)

The foregoing argument also shows $\beta_{k+1} = 0$ in (13); hence $\beta_k = \beta_{k+2} = 1$ from the coefficient constraints, and

$$\sum_{1}^{k} u_i + 1 = \sum_{1}^{k+m} \beta_i v_i \geq v_{k+2} + u_k + u_{k-2} + \ldots + u_{1,2} =$$

$$= v_{k+2} + u_{k+1} - 1 = v_{k+2} + \sum_{1}^{k-1} u_i + 1 \quad ,$$

or $u_k \geq v_{k+2}$. From Corollary 1, we infer (note $v_k = u_k$) that

(14) $$v_{k+2} < u_k \quad .$$

Now, consider the particular integer, $N = v_{k+2} + v_k + v_{k-2} + \ldots + v_{1,2}$, which is in this admissible form of (10) - (11). We have

(15) $$v_{k+2} + v_k + v_{k-2} + \ldots + v_{1,2} < u_k + (u_k + u_{k-2} + \ldots + u_{1,2}) =$$

$$= u_k + u_{k+1} - 1 = u_{k+2} - 1 \quad ,$$

or

$$N = v_{k+2} + v_k + v_{k-2} + \ldots + v_{1,2} \leq u_{k+2} - 2 = \sum_{1}^{k} v_i \quad .$$

Thus N also has a representation in admissible form involving at most the first k members of the sequence $\{v_i\}$, and uniqueness is contradicted.

The inequality $v_{k+1} > u_{k+1}$, is therefore untenable and we have shown $v_{k+1} = u_{k+1}$. The theorem then follows immediately by induction.

Thus, the dual unique representation property (Property D) is a property enjoyed only by the Fibonacci numbers and is therefore sufficient to characterize the sequence $\{u_i\}$.

Acknowledgement: I would like to acknowledge the contribution of Professor Verner E. Hoggatt, Jr., whose catalytic comments led to the theorems of this paper. See also a paper by H. H. Ferns [3] this issue.

REFERENCES

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