ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Mathematics Department, University of Santa Clara, Santa Clara, California. Any problem believed to be new in the area of recurrent sequences and any new approaches to existing problems will be welcomed. The proposer should submit each problem with solution in legible form, preferably typed in double spacing with name and address of the proposer as a heading.

Solutions to problems listed below should be submitted on separate signed sheets within two months of publication.

B-58 *Proposed by Sidney Kravitz, Dover, New Jersey*

Show that no Fibonacci number other than 1, 2, or 3 is equal to a Lucas number.

B-59 *Proposed by Brother U. Alfred, St. Mary's College, California*

Show that the volume of a truncated right circular cone of slant height F_n with F_{n-1} and F_{n+1} the diameters of the bases is

$$\sqrt{3}\,\pi(F_{n+1}^3 - F_{n-1}^3)/24 .$$

B-60 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that $L_{2n}L_{2n+2} - 5F_{2n+1}^2 = 1$, where F_n and L_n are the n-th Fibonacci number and Lucas number, respectively.

B-61 *Proposed by J.A.H. Hunter, Toronto, Ontario*

Define a sequence $U_1, U_2, \ldots$ by $U_1 = 3$ and

$$U_n = U_{n-1} + n^2 + n + 1 \quad \text{for } n > 1 .$$

Prove that $U_n \equiv 0 \pmod n$ if $n \not\equiv 0 \pmod 3$.

B-62 *Proposed by Brother U. Alfred, St. Mary's College, California*

Prove that a Fibonacci number with odd subscript cannot be represented as the sum of squares of two Fibonacci numbers in more than one way.

B-63 *An old problem whose source is unknown, suggested by Sidney Kravitz, Dover, New Jersey*

In ΔABC let sides AB and AC be equal. Let there be a point D on side AB such that AD = CD = BC. Show that

$$2\cos \measuredangle A = AB/BC = (1 + \sqrt{5})/2 \quad ,$$

the golden mean.

SOLUTIONS

A BOUND ON BOUNDED FIBONACCI NUMBERS

B-44 *Proposed by Douglas Lind, Falls Church, Virginia*

Prove that for every positive integer k there are no more than n Fibonacci numbers between n^k and n^{k+1}.

Solution by the proposer.

Assume the maximum,

$$(1) \qquad n^k < F_{r+1},\ F_{r+2},\ \ldots,\ F_{r+n} < n^{k+1} \quad .$$

Now

$$\sum_{j=1}^{n-1} F_{r+j} = \sum_{j=1}^{r+n-1} F_j - \sum_{j=1}^{r} F_j$$

$$= F_{r+n+1} - F_{r+2} \quad .$$

But by (1),

$$\sum_{j=1}^{n-1} F_{r+j} + F_{r+2} > n \cdot n^k$$

and hence

$$F_{r+n+1} > n^{k+1} \quad ,$$

thus proving the proposition.

ANOTHER SUM

B-45 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Let H_n be the n-th generalized Fibonacci number, i.e., let H_1 and H_2 be arbitrary and $H_{n+2} = H_{n+1} + H_n$ for $n > 0$. Show that $nH_1 + (n-1)H_2 + (n-2)H_3 + \ldots + H_n = H_{n+4} - (n+2)H_2 - H_1$.

Solution by David Zeitlin, Minneapolis, Minnesota.

In B-20 (see Fibonacci Quarterly, 2(1964) p. 77), it was shown that

$$\sum_{j=1}^{n} H_j = H_{n+2} - H_2 \quad .$$

In B-40 (see Fibonacci Quarterly, 2(1964), p. 155), Wall proposed that

$$\sum_{j=1}^{n} jH_j = (n+1)H_{n+2} - H_{n+4} + H_1 + H_2 \quad .$$

Thus, the desired sum

$$\sum_{j=1}^{n} \left[(n+1) - j\right] H_j = (n+1) \sum_{j=1}^{n} H_j - \sum_{j=1}^{n} jH_j$$

$$= \left[(n+1)H_{n+2} - (n+1)H_2\right] - \left[(n+1)H_{n+2} - H_{n+4} + H_1 + H_2\right]$$

$$= H_{n+4} - (n+2)H_2 - H_1 \quad .$$

Also solved by Douglas Lind, Kenneth E. Newcomer, Farid K. Shuayto, Sheryl B. Tadlock, Howard L. Walton, Charles Zeigenfus, and the proposer.

A CONTINUANT DETERMINANT

B-46 *Proposed by C.A. Church, Jr., Duke University, Durham, North Carolina*

Evaluate the n-th order determinant

$$D_n = \begin{vmatrix} a+b & ab & 0 & 0 & \ldots \\ 1 & a+b & ab & 0 & \ldots \\ 0 & 1 & a+b & ab & \ldots \\ 0 & 0 & 1 & a+b & \ldots \\ \ldots & & & & \\ \ldots & & & & \\ \ldots & & & & \end{vmatrix} .$$