ON THE REPRESENTATION OF INTEGERS AS SUMS OF DISTINCT FIBONACCI NUMBERS

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This paper gives an elementary discussion of the problem of expressing an arbitrary positive integer as a sum of distinct Fibonacci numbers. The recursive relation

$$(1) \qquad F_{n+2} = F_{n+1} + F_n$$

together with $F_1 = F_2 = 1$ is used as the definition of Fibonacci numbers. No use will be made of F_1 in any representation.

As an example consider the integer 29. It may be expressed as a sum of Fibonacci numbers in the following ways:

$$29 = F_8+F_6 = F_8+F_5+F_4 = F_8+F_5+F_3+F_2$$

$$= F_7+F_6+F_5+F_4 = F_7+F_6+F_5+F_3+F_2$$

From this example it immediately becomes apparent that we shall need to impose some "ground rules" if we are to differentiate between the various types of representations. This leads us to the following definitions.

A representation will be called minimal if it contains no two consecutive Fibonacci numbers.

A representation is said to be maximal if no two consecutive Fibonacci numbers F_i and F_{i+1} are omitted from the representation, where $F_2 \leq F_i < F_{i+1} \leq F_n$ and F_n is the largest Fibonacci number involved in the representation.

Thus $F_8 + F_6$ is a minimal representation of the integer 29 while $F_7+F_6+F_5+F_3+F_2$ is a maximal representation.

It follows that a maximal (minimal) representation may be transformed into a minimal (maximal) one by the application or repeated application of (1).

We shall first restrict our attention to minimal representations.

As an illustrative example of minimal representations we consider the representations of all integers N such that

$$F_7 \leq N < F_8$$

Thus N may be any one of the eight integers 13, 14, 15, 16, 17,18, 19, or 20. Now the smallest integer in this set, namely 13, cannot be represented by the Fibonacci numbers F_2, F_3, F_4, F_5 and F_6, since the largest integer under the minimal representation rule which they can represent is

$$F_6+F_4+F_2 = 12$$

Hence to represent all integers of this set requires F_2, F_3, F_4, F_5, F_6 and F_7.

By trial we obtain the following representations

$$13 = F_7;\ 14 = F_7+F_2;\ 15 = F_7+F_3;\ 16 = F_7+F_4;$$

$$17 = F_7+F_4+F_2;\ 18 = F_7+F_5;\ 19 = F_7+F_5+F_2;$$

$$20 = F_7+F_5+F_3 \quad .$$

One of these integers, namely 13, requires only one Fibonacci number to represent it. Four of them, namely, 14, 15, 16 and 18 require two and three of them 17, 19, and 20 require three.

Let $U_{n,m}$ denote the number of integers N in the range $F_n \leq N < F_{n+1}$ which require m Fibonacci numbers to represent them.

Then

$$U_{7,1} = 1;\ U_{7,2} = 4;\ U_{7,3} = 3 \quad .$$

It is also evident that

$$U_{7,1} + U_{7,2} + U_{7,3} = F_8 - F_7 = F_6 = 8 \quad .$$

Now it is known (1) that

$$U_{n,m} = 0, \text{ if } m > \left[\tfrac{n}{2}\right]$$

Thus we may write

$$\sum_{i=1}^{n} U_{n,i} = F_{n+1} - F_n = F_{n-1} \quad .$$

Table I gives values of $U_{n,m}$ for $n = 1, 2, 3, \ldots, 8$; $m = 1, 2, 3, \ldots, 4$.

We now discuss some properties of the function $U_{n,m}$.

Consider integers P, Q and R, defined by the following relations

$$F_n \leq P < F_{n+1};\ F_{n-1} \leq Q < F_n;\ F_{n-2} \leq R < F_{n-1}$$

Thus

$$P = F_n + p, \quad p = 0, 1, 2, \ldots, F_{n-1} - 1 \tag{2}$$

$$Q = F_{n-1} + q, \quad q = 0, 1, 2, \ldots, F_{n-2} - 1 \tag{3}$$

$$R = F_{n-2} + r, \quad r = 0, 1, 2, \ldots, F_{n-3} - 1 \tag{4}$$

We arrange the integers P in two sets (A) and (B) as follows.

(A) $$P = F_n + p_1, \quad p_1 = 0, 1, 2, \ldots, F_{n-2} - 1$$

(B) $$P = F_n + p_2, \quad p_2 = F_{n-2}, F_{n-2}+1, F_{n-2}+2, \ldots, F_{n-2} + (F_{n-1} - F_{n-2} - 1)$$

$$= F_{n-2} + r, \quad r = 0, 1, 2, \ldots, F_{n-3} - 1$$

If k is a positive integer, (1) implies that

$$F_n + k = F_{n-1} + k + F_{n-2}$$

Hence for the set (A)

$$F_n + p_1 = F_{n-1} + p_1 + F_{n-2}$$
$$P = F_{n-1} + q + F_{n-2}$$
$$P = F_n + q\ .$$

Comparing the last equation with (2) and (3) we see that the representation of an integer Q may be converted into a representation of an integer P of the set (A) by replacing F_{n-1} in the former by F_n.

By this operation we may derive the representations of F_{n-2} of the integers P from the representations of the integers Q. Derivation of the representations of P in this manner leaves the number of Fibonacci numbers unchanged.

We have now to consider the integers P in the set (B).

We have

$$P = F_n + p_2, \quad p_2 = F_{n-2},\ F_{n-2}+1,\ F_{n-2}+2, \ldots,\ F_{n-2}+(F_{n-1}-F_{n-2}-1)$$

$$= F_n + F_{n-2} + r, \quad r = 0, 1, 2, \ldots,\ F_{n-3}-1$$

$$= F_n + R \quad \text{by (4)}$$

The last result implies that the representations of the integers P in the set (B) may be derived from the representations of the integers R by adding F_n to each of the latter. This operation increases by one the number of the F_i in the representation of P over that of R from which it is derived.

By these two operations the representations of the F_{n-1} integers in $F_n \leq P < F_{n+1}$ can be derived from the representations of the F_{n-2} integers in $F_{n-1} \leq Q < F_n$ and the F_{n-3} integers in $F_{n-2} \leq Q < F_{n-1}$.

The following equations follow from the above discussion:

$$u_{n,m} = u_{n-1,m} + u_{n-2,m-1} \qquad (n > 2, m > 1)$$

$$(5) \qquad u_{n,1} = 1$$

$$u_{n,m} = 0 \quad \text{for } 2m > n.$$

These equations indicate that the $u_{n,m}$ may be related to the binomial coefficients $\binom{r}{k}$, which have the following properties:

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$$

$$\binom{r}{0} = 1$$

$$\binom{r}{k} = 0 \quad \text{for } k > r.$$

Letting $U_{n,m} = \binom{n-m-1}{m-1}$, these relations for the $\binom{r}{k}$ become the relations (5) with the $U_{n,m}$ substituted for the $u_{n,m}$. Since (5) makes it possible to calculate any $u_{n,m}$ with $n > 2$ and $m > 1$, these relations characterize the $u_{n,m}$ and so $u_{n,m} = U_{n,m} = \binom{n-m-1}{m-1}$ for $n > 2$ and $m > 1$.

The reader is referred to the paper "A Combinational Problem" by Lafer and Long in the November 1962 issue of the American Mathematical Monthly for an expository account of the inductive and deductive aspects of a similar problem. [3]

The proof of this is left to the reader.

We turn now to a discussion of maximal representation of integers as sums of Fibonacci numbers. In this discussion we shall use a different technique, one that could have been used equally well in the discussion of minimal representations.

As an example we consider the integers N such that $F_7-1 \leq N < F_8-1$. These are, 13, 14, 15, 16, 17, 18, 19 and 20. The reason for using the range $F_7-1 \leq N < F_{8-1}$ instead of $F_7 \leq N < F_8$ will become evident later.

Bearing in mind the definition of maximal representation we derive the following representations

$$12 = F_6+F_4+F_2;\ 13 = F_6+F_4+F_3;\ 14 = F_6+F_4+F_3+F_2;$$

$$15 = F_6+F_5+F_3;\ 16 = F_6+F_5+F_3+F_2;\ 17 = F_6+F_5+F_4+F_2;$$

$$18 = F_6+F_5+F_4+F_3;\ 19 = F_6+F_5+F_4+F_3+F_2 \quad .$$

These eight representations may be written compactly in the following form.

$$\begin{array}{rcccccccl}
 & & & F_6 & F_5 & F_4 & F_3 & F_2 & \\
12 & = & (& 1 & 0 & 1 & 0 & 1 &) \\
13 & = & (& 1 & 0 & 1 & 1 & 0 &) \\
14 & = & (& 1 & 0 & 1 & 1 & 1 &) \\
15 & = & (& 1 & 1 & 0 & 1 & 0 &) \\
16 & = & (& 1 & 1 & 0 & 1 & 1 &) \\
17 & = & (& 1 & 1 & 1 & 0 & 1 &) \\
18 & = & (& 1 & 1 & 1 & 1 & 0 &) \\
19 & = & (& 1 & 1 & 1 & 1 & 1 &)
\end{array}$$

In this display we have used the binary digits in conjunction with Fibonacci numbers denoting place values. It should be noticed that with this scheme two zeros cannot be in adjacent places in maximal representation. For example (...100...) must be replaced by (...011...) since $F_i = F_{i-1} + F_{i-2}$. Also the Fibonacci numbers denoting the positional values are arranged in ascending order from the right to left beginning with F_2.

We now consider the general case. Let N be an integer defined by

$$F_n - 1 \leq N < F_{n+1} - 1$$

Let $V_{n,m}$ denote the number of integers N in this interval which require m Fibonacci numbers to represent them in maximal representation.

Thus for the illustrative example given above

$$V_{7,3} = 3; \; V_{7,4} = 4; \; V_{7,5} = 1$$

Also

$$V_{7,3} + F_{7,4} + V_{7,5} = F_8 - F_7 = F_6 = 8$$

The largest integer in the interval $F_n - 1 \leq N < F_{n+1} - 1$ is $F_{n+1} - 2$ and since (2)

$$\sum_{i=2}^{n-1} F_i = F_{n+2} - 2$$

it follows that

$$F_{n+1} - 2 = (111\ldots 11) \quad \text{(n-2 digits)}$$

in which no zeros appear and in which the left hand positional value is F_{n-1}. This explains the reason for taking the upper bound of N to be $F_{n+1} - 1$ instead of F_{n+1}.

The smallest integer in the range in question is $F_n - 1$ and since

$$\sum_{i=2}^{n-2} F_i = F_n - 2 < F_n - 1$$

it follows that there must be a "one" in the first (left hand position) defined by F_{n-1}. Further since (2)

$$F_2 + F_4 + F_6 + \ldots + F_n = F_{n+1} \quad (n \text{ even})$$

$$F_3 + F_5 + F_7 + \ldots + F_n = F_{n+1} \quad (n \text{ odd})$$

it follows that the smallest integer in the range in question is indicated by

$$(1010\ldots10) \quad \text{or} \quad (1010\ldots101)$$

according as n is odd or even.

From these observations we conclude that

$$V_{n,m} = 0 \quad \text{if} \begin{cases} m > n-2 & \text{or} \quad n < m+2 \\ m < \left[\frac{n-1}{2}\right] & \text{or} \quad n > 2(m+1) \end{cases}$$

$$\sum_{i=\left[\frac{n-1}{2}\right]}^{n-2} V_{n,i} = F_{n+1} - F_n = F_{n-1}$$

Table II gives values of $V_{n,m}$ for $n = 2, 3, \ldots 12; m = 1, 2, \ldots, 10$.

We now establish the recursive relation

(6) $$V_{n,m} = V_{n-1,m-1} + V_{n-2,m-1}$$

Consider integers P, Q and R defined by

$$F_n - 1 \leq P < F_{n+1} - 1$$

$$F_{n-1} - 1 \leq Q < F_n - 1$$

$$F_{n-2} - 1 \leq R < F_{n-1} - 1$$

The Fibonacci positional representation of the integers Q are of the type

$$\begin{array}{rcccccc} & F_{n-2} & F_{n-3} & F_{n-4} & \cdots & F_2 & \\ Q = (& 1 & a & b & \ldots & c &) \end{array}$$

Adding F_{n-1} to each integer Q will produce F_{n-2} integers all of which will be within the interval

$$F_{n-1}+F_{n-1}-1 \leq Q + F_{n-1} < F_{n-1}+F_n-1 \quad .$$

This is equivalent to

$$F_{n+1}-F_n+F_{n-1}-1 \leq Q + F_{n-1} < F_{n+1}-1$$

$$F_{n+1}-F_{n-1}-F_{n-2}+F_{n-1}-1 \leq Q + F_{n-1} < F_{n+1}-1$$

$$F_{n+1}-F_{n-2}-1 \leq Q + F_{n-1} < F_{n+1}-1$$

$$F_n+F_{n-3}-1 \leq Q + F_{n-1} < F_{n+1}-1$$

These F_{n-2} integers $Q + F_{n-1}$ are all in the interval $F_n-1 \leq P < F_{n+1}-1$. Their positional representation takes the form

$$\begin{array}{rccccccc} & F_{n-1} & F_{n-2} & F_{n-3} & F_{n-4} & \cdots & F_2 & \\ Q + F_{n-1} = (& 1 & 1 & a & b & \ldots & c &) \end{array}$$

Hence the representations of F_{n-2} of the integers P may be derived from the integers Q by creating an additional position defined as F_{n-1}.

The F_{n-3} integers R have positional representations of the form

$$\begin{array}{rcccccc} & F_{n-3} & F_{n-4} & F_{n-5} & \cdots & F_2 & \\ R = (& 1 & d & e & \ldots & f &) \end{array}$$

Adding F_{n-1} to each of these F_{n-3} integers will result in integers all in the interval

$$F_{n-1}+F_{n-2}-1 \leq R + F_{n-1} < F_{n-1}+F_{n-1}-1$$

$$F_n-1 \leq R + F_{n-1} < F_{n+1}-F_{n-2}-1 \quad .$$

That is, these F_{n-3} integers are all within the interval $F_n - 1 \leq P < F_{n+1} - 1$. Each of them will have representations of the form

$$\begin{array}{rcccccccc} & & F_{n-1} & F_{n-2} & F_{n-3} & F_{n-4} & F_{n-5} & \cdots & F_2 \\ R + F_{n-1} & = (& 1 & 0 & 1 & d & e & \ldots & f\ ) \end{array}$$

Hence the representations of F_{n-3} of the integers P may be obtained from the representations of R by adding on the left two positional values namely F_{n-1} and F_{n-2}.

Since the first operation results in a representation which has a "one" in the second (from the left) place while the second operation gives a representation with a zero in that place the two representations are disjoint. Thus there is no overlapping and all integers P are accounted for by these two operations.

This completes the proof of (6).

It is readily verified that

$$V_{n,m} = \binom{m}{n-m-2} \tag{7}$$

satisfies the recursive relation (6).

From (7) we find that

$$\sum_{i=m+2}^{2(m+1)} V_{i,m} = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \ldots + \binom{m}{m}$$

Also from the paragraph following (5) and (7) we see that

$$V_{n,m} = U_{n,n-m-1} \quad .$$

REFERENCES

1. P. Lafer, "Exploring the Fibonacci Representation of Integers," The Fibonacci Quarterly, April 1964, p. 114.
2. Cf. N. N. Vorob'ev, Fibonacci Numbers, New York, 1961, pp. 6, 7.
3. P. Lafer and C. T. Long, "A Combinatorial Problem," <u>The American Mathematical Monthly</u>, Nov. 1962, pp. 876-883.

Table I

Values of $U_{n,m}$ $n = 1, 2, \ldots, 8$ $m = 1, 2, \ldots, 4$

	n \ m	1	2	3	4
$F_1 \le N < F_2$	1	0	0	0	0
$F_2 \le N < F_3$	2	1	0	0	0
$F_3 \le N < F_4$	3	1	0	0	0
$F_4 \le N < F_5$	4	1	1	0	0
$F_5 \le N < F_6$	5	1	2	0	0
$F_6 \le N < F_7$	6	1	3	1	0
$F_7 \le N < F_8$	7	1	4	3	0
$F_8 \le N < F_9$	8	1	5	6	1

Table II

Values of $V_{n,m}$ $n = 2, 3, 4, \ldots, 12;$ $m = 1, 2, 3, \ldots, 10$

	n \ m	1	2	3	4	5	6	7	8	9	10
$F_2-1 \le N < F_3-1$	2	0	0	0	0	0	0	0	0	0	0
$F_3-1 \le N < F_4-1$	3	1	0	0	0	0	0	0	0	0	0
$F_4-1 \le N < F_5-1$	4	1	1	0	0	0	0	0	0	0	0
$F_5-1 \le N < F_6-1$	5	0	2	1	0	0	0	0	0	0	0
$F_6-1 \le N < F_7-1$	6	0	1	3	1	0	0	0	0	0	0
$F_7-1 \le N < F_8-1$	7	0	0	3	4	1	0	0	0	0	0
$F_8-1 \le N < F_9-1$	8	0	0	1	6	5	1	0	0	0	0
$F_9-1 \le N < F_{10}-1$	9	0	0	0	4	10	6	1	0	0	0
$F_{10}-1 \le N < F_{11}-1$	10	0	0	0	1	10	15	7	1	0	0
$F_{11}-1 \le N < F_{12}-1$	11	0	0	0	0	5	20	21	8	1	0
$F_{12}-1 \le N < F_{13}-1$	12	0	0	0	0	1	15	35	28	9	1

N. B. The entries in the vertical columns are rows of PASCAL's arithmetic triangle so that the table may be easily extended.

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