EXPLORING SCALENE FIBONACCI POLYGONS

Proposed by BROTHER U. ALFRED on page 60, October 1963

"The Fibonacci Quarterly"
C.B.A. Peck

The sequence of Fibonacci numbers may be defined by

\[ F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geq 2. \]

We describe a way of deciding when a set \( S \) of \( m \) distinct numbers drawn from the sequence \( F_2, F_3, F_4, \ldots \) corresponds to the sides of some plane polygon with \( m \) sides. If they do we call \( S \) a (scalene Fibonacci) polygon, for short.

To prove the result, we find it convenient to use the following identities, easily proved from (1) by induction on \( k \) and \( n \), respectively:

\[ F_n = F_{n-2k} + \sum_{i=1}^{k} F_{n-2i+1} \quad \text{for} \quad n \geq 4 \quad \text{and} \quad 0 < 2k < n, \]

\[ F_n > \sum_{i=2}^{n-2} F_i \quad \text{for} \quad n \geq 1 \quad \text{(the sum is zero if} \quad n = 1, 2, 3). \]

Suppose once and for all that \( F_n \) is the largest number in \( S \). If we denote by \( S(n,k) \) the set of numbers appearing in (2), then \( S \) is a polygon if and only if it properly contains some \( S(n,k) \). If it equals \( S(n,k) \) for some \( k \) we call it a degenerate polygon.

Proof: If \( F_{n-1} \notin S \), then by (2) \( S \) contains no \( S(n,k) \). By (3) \( F_n \) exceeds the sum of the other numbers in \( S \), which shows that \( S \) is not even a degenerate polygon. Now suppose that \( F_{n-1} \notin S \) (so that \( n \geq 3 \)) and proceed downward through the sequence in (1), starting with \( F_{n-1} \) and stopping short of \( F_1 \). The numbers alternate in and out of \( S \) until one of two things happens.

1. \( S \) is found to contain no \( S(n,k) \), either because the alternation stops at an adjacent pair not in \( S \), say \( F_{n-2j} \) \( F_{n-2j-1} \) with \( n-2j-1 \geq 2 \), or continues to the bottom (here we set \( n-2j-1 = 1 \) or 0 according as \( n \) is even or odd). Then every number in \( S \) other than \( F_n \) occurs in either

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The first sum \( < F_{n-2j} \) by (3), whence the sum of each \( < F_n \) by (2).
Thus \( S \) is again not even a degenerate polygon.

2. The alternation stops with an adjacent pair in \( S \), say \( F_{n-2k+1}^1 \).
\( F_{n-2k} \) with \( n-2k \geq 2 \), so that \( S(n,k) \) is in \( S \). Then (2) shows that
\( S \) is a (degenerate) polygon if there are (no) numbers in \( S \) besides
those in \( S(n,k) \), on the grounds that \( F_n \), the sum of the other num-
bers in \( S \).

Could two sets of numbers drawn from \( F_2, F_3, F_4, \ldots \) be pro-
portional to the lengths of the sides of a single polygon? This is not
possible, at any rate, when the numbers in each set are distinct, for
suppose that we did have two scalene Fibonacci polygons with largest
sides \( F_n \) and \( F_N \) \( (N > n) \) proportional to a third polygon, hence to
each other, say in the ratio \( P > 1 \). We have just seen that if \( F_n \) is
the largest number in such a set, then \( F_{n-1} \) must belong to it. Since
the largest and second largest sides correspond, we have \( PF_n = F_N \)
and \( PF_{n-1} = F_{N-1} \). By (1), we have, then, \( PF_{n-2} = F_{N-2} \) and \( n-2 \)
further applications of (1) yield finally \( PF_0 = F_{N-n} \). By (1), the l.h.s.
is zero and the r.h.s. positive, which is absurd.

An interesting exercise is to use this argument (with suitable
amplification of the last sentence) on any two Fibonacci polygons such
that in at least one of them there are numbers whose subscripts differ
by only one or two. We need something stronger for such polygons as
\( F_n', F_{n-3}', F_{n-3}', F_{n-3}, F_{n-3}', F_{n-3} \).
The generalization of (2) which seems to be called for is some
characterization of the coefficients in inequalities of the form

\[
\sum_{i=2}^{n} a_i F_i \leq F_n
\]

where the \( a_i \)'s are nonnegative integers.