PRODUCTS OF ODDS

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The following are used in the proofs of the identities:

\[ F_n = \frac{\beta^n - a^n}{\beta - a}, \quad L_n = \beta^n + a^n, \quad (\beta a)^n = (-1)^n, \]

where

\[ \beta = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad a = \frac{1 - \sqrt{5}}{2} \]

1. \( F_{2k+1} F_{2j+1} = F_{k+j+1}^2 + F_{k-j}^2 \)

\[ F_{k+j+1}^2 + F_{k-j}^2 = \left[ \frac{\beta^{k+j+1} - a^{k+j+1}}{\beta - a} \right]^2 + \left[ \frac{\beta^{k-j} - a^{k-j}}{\beta - a} \right]^2 \]

\[ = \frac{\beta^2 k + 2 j + 2}{\beta - a} + a^2 k + 2 j + 2 - 2(\beta a)^{k+j+1} - 2(\beta a)^{-k-j} + a 2 k - 2 j + a 2 k - 2 j} \]

\[ = \frac{\beta^2 k + 1 (\beta^2 j + 1) + a 2 k + 1 (a^2 j + 1 - a^2 j - 1)}{\beta - a} \]

Recalling that \( \beta^{-2j-1} = (-1)^{-2j-1} a^{2j+1} = -a^{2j+1} \) and

\[ a^{-2j-1} = (-1)^{-2j-1} \beta^{2j+1} = -\beta^{2j+1} \]

and that the last term has the value of 0, the above expression becomes

\[ F_{k+j+1}^2 + F_{k-j}^2 = \frac{\beta^2 k + 1 (\beta^2 j + 1) - a^2 j + 1 - a^2 j + 1}{\beta - a} \]

\[ = \frac{\beta^2 k + 1 (\beta^2 j + 1) - a^2 j + 1}{\beta - a} \]

But the right-hand side is of the form \( F_{2k+1} F_{2j+1} \). Therefore,

\[ F_{k+j+1}^2 + F_{k-j}^2 = F_{2k+1} F_{2j+1} \]

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2. \( L_{2k+1} \cdot L_{2j+1} = \frac{L^2}{k+j+1} - \frac{L^2}{k-j} + 4(-1)^{k-j} \)

\[
L_{2k+1} \cdot L_{2j+1} = (\beta^{2k+1} + a^{2k+1})(\beta^{2j+1} + a^{2j+1})
\]

\[
= \beta^{2k+2j+2} + a^{2k+2j+2} + \beta^{2k+1}a^{2j+1} + a^{2k+1} \beta^{2j+1}
\]

\[
= \beta^{2k+2j+2} + a^{2k+2j+2} + \beta^{k+j+1}a^{k+j+1} + (\beta^{k-j}a^{j+k} + a^k \beta^{j+k})
\]

Observing that

\[
\beta^{k-j}a^{j-k} = (-1)^{j-k}\beta^{j-k}a^{k-j} = (-1)^{j-k}\beta^{2k-2j}
\]

and

\[
a^{k-j}\beta^{j-k} = (-1)^{j-k}a^{j-k}a^{k-j} = (-1)^{j-k}a^2k-2j
\]

and adding and subtracting \( 2\beta^{k+j+1}a^{k+j+1} \) the above expression becomes

\[
L_{2k+1} \cdot L_{2j+1} = \beta^{2k+2j+2} + a^{2k+2j+2} + 2\beta^{k+j+1}a^{k+j+1} - 2\beta^{k-j-1}a^{k-j+1}
\]

\[
+ (\beta a)^{k+j+1}((-1)^{j-k}\beta^{2k-2j} + (-1)^{j-k}a^{2k-2j})
\]

\[
= \beta^{2k+2j+2} + a^{2k+2j+2} + 2\beta^{k+j+1}a^{k+j+1} + a^{2k+2j+2}
\]

\[
+ (\beta a)^{k+j+1}(-2 + (-1)^{j-k}\beta^{2k-2j} + (-1)^{j-k}a^{2k-2j})
\]

\[
= (\beta^{k+j-1} + a^{k+j+1})^2 + (-1)^{k+j+1}((-1)^{j-k}\beta^{2k-2j} + a^{2k-2j} - 2\beta^{k-j}a^{k-j})
\]

\[
= L^2_{k+j+1} + (-1)^{2j+1}(-2\beta^{k-j} + 2\beta^{k-j}a^{k-j} + a^{2k-2j} - 2\beta^{k-j}a^{k-j})
\]

Noting that

\[
-2(-1)^{k-j}(-1)^{2j+1} = -2(-1)^{k-j}(-1) = 2(-1)^{k-j}
\]

we have

\[
L_{2k+1} \cdot L_{2j+1} = \frac{L^2}{k+j+1} - \left[ (\beta^{k-j} + a^{k-j})^2 - 2(\beta a)^{k-j} \right] + 2(-1)^{k-j}
\]

\[
= L^2_{k+j+1} - (L^2_{k-j} - 2(-1)^{k-j}) + 2(-1)^{k-j}
\]

\[
= L^2_{k+j+1} - L^2_{k-j} + 2(-1)^{k-j} + 2(-1)^{k-j}
\]

Therefore,

\[
L_{2k+1} \cdot L_{2j+1} = \frac{L^2}{k+j+1} - \frac{L^2}{k-j} + 4(-1)^{k-j}
\]
By using the identity \( \frac{L_n^2}{n} - 5 \frac{F_n^2}{n} = 4(-1)^n \) (Vol. 1, No. 1, p. 66, this Quarterly), it can easily be shown that

\[
\frac{L_{k+j+1}^2}{k+j+1} - \frac{L_{k-j}^2}{k-j} + 4(-1)^{k-j} = \frac{L_{k+j+1}^2}{k+j+1} - 5 \frac{F_{k-j}^2}{k-j} = 5 \frac{F_{k+j+1}^2}{k+j+1} - \frac{L_{k-j}^2}{k-j}.
\]

Thus, we have proofs of the following Fibonacci identity and the analogous Lucas identities for products of odds:

1. \( F_{2k+1} F_{2j+1} = F_{k+j+1}^2 + F_{k-j}^2 \)
2. \( L_{2k+1} L_{2j+1} = L_{k+j+1}^2 - L_{k-j}^2 + 4(-1)^{k-j} \)
3. \( L_{2k+1} L_{2j+1} = L_{k+j+1}^2 - 5 F_{k-j}^2 \)
4. \( L_{2k+1} L_{2j+1} = 5 F_{k+j+1}^2 - L_{k-j}^2 \).

These four identities correspond closely to those given for products of evens in this Quarterly, Vol. 2, No. 1, p. 78.

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