

PRODUCTS OF ODDS

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The following are used in the proofs of the identities:

$$F_n = \frac{\beta^n - a^n}{\beta - a}, \quad L_n = \beta^n + a^n, \quad (\beta a)^n = (-1)^n,$$

where

$$\beta = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad a = \frac{1 - \sqrt{5}}{2}$$

$$1. \quad F_{2k+1} F_{2j+1} = F_{k+j+1}^2 + F_{k-j}^2$$

$$\begin{aligned} F_{k+j+1}^2 + F_{k-j}^2 &= \left[\frac{\beta^{k+j+1} - a^{k+j+1}}{\beta - a} \right]^2 + \left[\frac{\beta^{k-j} - a^{k-j}}{\beta - a} \right]^2 \\ &= \frac{\beta^{2k+2j+2} + a^{2k+2j+2} - 2(\beta a)^{k+j+1} - 2(\beta a)^{k-j} + \beta^{2k-2j} + a^{2k-2j}}{(\beta - a)^2} \\ &= \frac{\beta^{2k+1}(\beta^{2j+1} + \beta^{-2j-1}) + a^{2k+1}(a^{2j+1} + a^{-2j-1})}{(\beta - a)^2} \\ &\quad - \frac{2(\beta a)^{k+j}(\beta a + (\beta a)^{-2j})}{(\beta - a)^2} \end{aligned}$$

Recalling that $\beta^{-2j-1} = (-1)^{-2j-1} a^{2j+1} = -a^{2j+1}$

and

$$a^{-2j-1} = (-1)^{-2j-1} \beta^{2j+1} = -\beta^{2j+1}$$

and that the last term has the value of 0, the above expression becomes

$$\begin{aligned} F_{k+j+1}^2 + F_{k-j}^2 &= \frac{\beta^{2k+1}(\beta^{2j+1} - a^{2j+1}) - a^{2k+1}(\beta^{2j+1} - a^{2j+1})}{(\beta - a)^2} \\ &= \frac{(\beta^{2k+1} - a^{2k+1})(\beta^{2j+1} - a^{2j+1})}{(\beta - a)(\beta - a)} \end{aligned}$$

But the right-hand side is of the form $F_{2k+1} F_{2j+1}$. Therefore,

$$F_{k+j+1}^2 + F_{k-j}^2 = F_{2k+1} F_{2j+1}$$

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$$2. L_{2k+1} L_{2j+1} = L_{k+j+1}^2 - L_{k-j}^2 + 4(-1)^{k-j}$$

$$\begin{aligned} L_{2k+1} L_{2j+1} &= (\beta^{2k+1} + \alpha^{2k+1})(\beta^{2j+1} + \alpha^{2j+1}) \\ &= \beta^{2k+2j+2} + \alpha^{2k+2j+2} + \beta^{2k+1} \alpha^{2j+1} + \alpha^{2k+1} \beta^{2j+1} \\ &= \beta^{2k+2j+2} + \alpha^{2k+2j+2} + \beta^{k+j+1} \alpha^{k+j+1} (\beta^{k-j} \alpha^{j+k} + \alpha^{k-j} \beta^{j-k}) \end{aligned}$$

Observing that

$$\beta^{k-j} \alpha^{j-k} = (-1)^{j-k} \beta^{k-j} \beta^{k-j} = (-1)^{j-k} \beta^{2k-2j}$$

and

$$\alpha^{k-j} \beta^{j-k} = (-1)^{j-k} \alpha^{k-j} \alpha^{k-j} = (-1)^{j-k} \alpha^{2k-2j}$$

and adding and subtracting $2\beta^{k+j+1} \alpha^{k+j+1}$ the above expression becomes

$$\begin{aligned} L_{2k+1} L_{2j+1} &= \beta^{2k+2j+2} + \alpha^{2k+2j+2} + 2\beta^{k+j+1} \alpha^{k+j+1} - 2\beta^{k-j-1} \alpha^{k-j+1} \\ &\quad + (\beta \alpha)^{k+j+1} ((-1)^{j-k} \beta^{2k-2j} + (-1)^{j-k} \alpha^{2k-2j}) \\ &= \beta^{2k+2j+2} + 2\beta^{k+j+1} \alpha^{k+j+1} + \alpha^{2k+2j+2} \\ &\quad + (\beta \alpha)^{k+j+1} (-2 + (-1)^{j-k} \beta^{2k-2j} + (-1)^{j-k} \alpha^{2k-2j}) \\ &= (\beta^{k+j-1} + \alpha^{k+j+1})^2 + (-1)^{k+j+1} (-1)^{j-k} (\beta^{2k-2j} + \alpha^{2k-2j} - 2(-1)^{k-j}) \\ &= L_{k+j+1}^2 + (-1)^{2j+1} (\beta^{2k-2j} + 2\beta^{k-j} \alpha^{k-j} + \alpha^{2k-2j} - 2\beta^{k-j} \alpha^{k-j} \\ &\quad - 2(-1)^{k-j} (-1)^{2j+1}) \end{aligned}$$

$$\text{Noting that } -2(-1)^{k-j} (-1)^{2j+1} = -2(-1)^{k-j} (-1) = 2(-1)^{k-j}$$

we have

$$\begin{aligned} L_{2k+1} L_{2j+1} &= L_{k+j+1}^2 - [(\beta^{k-j} + \alpha^{k-j})^2 - 2(\beta \alpha)^{k-j}] + 2(-1)^{k-j} \\ &= L_{k+j+1}^2 - (L_{k-j}^2 - 2(-1)^{k-j}) + 2(-1)^{k-j} \\ &= L_{k+j+1}^2 - L_{k-j}^2 + 2(-1)^{k-j} + 2(-1)^{k-j} \end{aligned}$$

Therefore,

$$L_{2k+1} L_{2j+1} = L_{k+j+1}^2 - L_{k-j}^2 + 4(-1)^{k-j}$$

By using the identity $L_n^2 - 5F_n^2 = 4(-1)^n$ (Vol. 1, No. 1, p. 66, this Quarterly), it can easily be shown that

$$L_{k+j+1}^2 - L_{k-j}^2 + 4(-1)^{k-j} = L_{k+j+1}^2 - 5F_{k-j}^2 = 5F_{k+j+1}^2 - L_{k-j}^2 .$$

Thus, we have proofs of the following Fibonacci identity and the analogous Lucas identities for products of odds:

- (1) $F_{2k+1} F_{2j+1} = F_{k+j+1}^2 + F_{k-j}^2$
- (2) $L_{2k+1} L_{2j+1} = L_{k+j+1}^2 - L_{k-j}^2 + 4(-1)^{k-j}$
- (3) $L_{2k+1} L_{2j+1} = L_{k+j+1}^2 - 5F_{k-j}^2$
- (4) $L_{2k+1} L_{2j+1} = 5F_{k+j+1}^2 - L_{k-j}^2 .$

These four identities correspond closely to those given for products of evens in this Quarterly, Vol. 2, No. 1, p. 78.

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