PRODUCTS OF ODDS

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The following are used in the proofs of the identities:

$$F_n = \frac{\beta^n - \alpha^n}{\beta - \alpha}, \quad L_n = \beta^n + \alpha^n, \quad (\beta\alpha)^n = (-1)^n,$$

where

$$\beta = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \alpha = \frac{1 - \sqrt{5}}{2}$$

1. $F_{2k+1} F_{2j+1} = F^2_{k+j+1} + F^2_{k-j}$

$$F^2_{k+j+1} + F^2_{k-j} = \left[\frac{\beta^{k+j+1} - \alpha^{k+j+1}}{\beta - \alpha}\right]^2 + \left[\frac{\beta^{k-j} - \alpha^{k-j}}{\beta - \alpha}\right]^2$$

$$= \frac{\beta^{2k+2j+2} + \alpha^{2k+2j+2} - 2(\beta\alpha)^{k+j+1} - 2(\beta\alpha)^{k-j} + \beta^{2k-2j} + \alpha^{2k-2j}}{(\beta - \alpha)^2}$$

$$= \frac{\beta^{2k+1}(\beta^{2j+1} + \beta^{-2j-1}) + \alpha^{2k+1}(\alpha^{2j+1} + \alpha^{-2j-1})}{(\beta - \alpha)^2}$$

$$- \frac{2(\beta\alpha)^{k+j}(\beta\alpha + (\beta\alpha)^{-2j})}{(\beta - \alpha)^2}$$

Recalling that $\beta^{-2j-1} = (-1)^{-2j-1}\alpha^{2j+1} = -\alpha^{2j+1}$

and $\alpha^{-2j-1} = (-1)^{-2j-1}\beta^{2j+1} = -\beta^{2j+1}$

and that the last term has the value of 0, the above expression becomes

$$F^2_{k+j+1} + F^2_{k-j} = \frac{\beta^{2k+1}(\beta^{2j+1} - \alpha^{2j+1}) - \alpha^{2k+1}(\beta^{2j+1} - \alpha^{2j+1})}{(\beta - \alpha)^2}$$

$$= \frac{(\beta^{2k+1} - \alpha^{2k+1})(\beta^{2j+1} - \alpha^{2j+1})}{(\beta - \alpha)\qquad(\beta - \alpha)}$$

But the right-hand side is of the form $F_{2k+1}F_{2j+1}$. Therefore,

$$F^2_{k+j+1} + F^2_{k-j} = F_{2k+1}F_{2j+1} \quad .$$

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2. $L_{2k+1}L_{2j+1} = L^2_{k+j+1} - L^2_{k-j} + 4(-1)^{k-j}$

$$L_{2k+1}L_{2j+1} = (\beta^{2k+1} + \alpha^{2k+1})(\beta^{2j+1} + \alpha^{2j+1})$$

$$= \beta^{2k+2j+2} + \alpha^{2k+2j+2} + \beta^{2k+1}\alpha^{2j+1} + \alpha^{2k+1}\beta^{2j+1}$$

$$= \beta^{2k+2j+2} + \alpha^{2k+2j+2} + \beta^{k+j+1}\alpha^{k+j+1}(\beta^{k-j}\alpha^{j+k} + \alpha^{k-j}\beta^{j-k})$$

Observing that $\beta^{k-j}\alpha^{j-k} = (-1)^{j-k}\beta^{k-j}\beta^{k-j} = (-1)^{j-k}\beta^{2k-2j}$

and $\alpha^{k-j}\beta^{j-k} = (-1)^{j-k}\alpha^{k-j}\alpha^{k-j} = (-1)^{j-k}\alpha^{2k-2j}$

and adding and subtracting $2\beta^{k+j+1}\alpha^{k+j+1}$ the above expression becomes

$$L_{2k+1}L_{2j+1} = \beta^{2k+2j+2} + \alpha^{2k+2j+2} + 2\beta^{k+j+1}\alpha^{k+j+1} - 2\beta^{k-j-1}\alpha^{k-j+1}$$

$$+ (\beta\alpha)^{k+j+1}((-1)^{j-k}\beta^{2k-2j} + (-1)^{j-k}\alpha^{2k-2j})$$

$$= \beta^{2k+2j+2} + 2\beta^{k+j+1}\alpha^{k+j+1} + \alpha^{2k+2j+2}$$

$$+ (\beta\alpha)^{k+j+1}(-2 + (-1)^{j-k}\beta^{2k-2j} + (-1)^{j-k}\alpha^{2k-2j})$$

$$= (\beta^{k+j-1} + \alpha^{k+j+1})^2 + (-1)^{k+j+1}(-1)^{j-k}(\beta^{2k-2j} + \alpha^{2k-2j} - 2(-1)^{k-j})$$

$$= L^2_{k+j+1} + (-1)^{2j+1}(\beta^{2k-2j} + 2\beta^{k-j}\alpha^{k-j} + \alpha^{2k-2j} - 2\beta^{k-j}\alpha^{k-j})$$

$$- 2(-1)^{k-j}(-1)^{2j+1} .$$

Noting that $-2(-1)^{k-j}(-1)^{2j+1} = -2(-1)^{k-j}(-1) = 2(-1)^{k-j}$

we have

$$L_{2k+1}L_{2j+1} = L^2_{k+j+1} - \left[(\beta^{k-j} + \alpha^{k-j})^2 - 2(\beta\alpha)^{k-j}\right] + 2(-1)^{k-j}$$

$$= L^2_{k+j+1} - (L^2_{k-j} - 2(-1)^{k-j}) + 2(-1)^{k-j}$$

$$= L^2_{k+j+1} - L^2_{k-j} + 2(-1)^{k-j} + 2(-1)^{k-j} .$$

Therefore,

$$L_{2k+1}L_{2j+1} = L^2_{k+j+1} - L^2_{k-j} + 4(-1)^{k-j} .$$

By using the identity $L_n^2 - 5F_n^2 = 4(-1)^n$ (Vol. 1, No. 1, p. 66, this Quarterly), it can easily be shown that

$$L_{k+j+1}^2 - L_{k-j}^2 + 4(-1)^{k-j} = L_{k+j+1}^2 - 5F_{k-j}^2 = 5F_{k+j+1}^2 - L_{k-j}^2 .$$

Thus, we have proofs of the following Fibonacci identity and the analogous Lucas identities for products of odds:

(1) $$F_{2k+1}F_{2j+1} = F_{k+j+1}^2 + F_{k-j}^2$$

(2) $$L_{2k+1}L_{2j+1} = L_{k+j+1}^2 - L_{k-j}^2 + 4(-1)^{k-j}$$

(3) $$L_{2k+1}L_{2j+1} = L_{k+j+1}^2 - 5F_{k-j}^2$$

(4) $$L_{2k+1}L_{2j+1} = 5F_{k+j+1}^2 - L_{k-j}^2 .$$

These four identities correspond closely to those given for products of evens in this Quarterly, Vol. 2, No. 1, p. 78.

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