MYSTERY PUZZLER AND PHI

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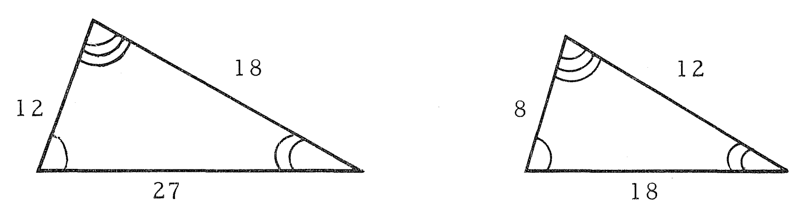
A problem proposed by Professor Hoggatt is as follows: Does there exist a pair of triangles which have five of their six parts equal but which are not congruent? (Here the six parts are the three sides and the three angles.) The initial impulsive answer is no! The problem also appears in [1] as well as in the MATH LOG.

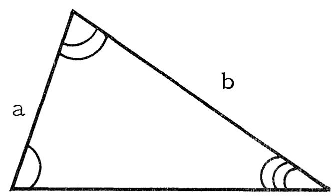
I have taken some time to work on the problem you suggested. I think you will agree that the solution I have is interesting. One problem, as you have stated it, is posed in a high school geometry text entitled, "Geometry" by Moise and Downs, published by Addison Wesley Company, (page 369).

In their solution key, they gave one possible pair of triangles that work:



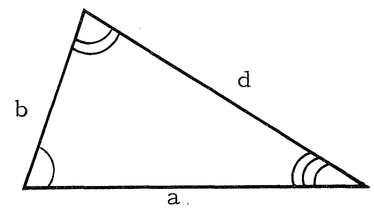
I discovered this after I solved the problem myself. But the above solution does not do justice to the problem at all, since my old friend τ is really the key to the solution. Note: Golden Mean = ϕ = τ in what follows.

I attacked the problem as follows: First, the five congruent parts cannot contain all three sides, since the triangles would then be congruent. Therefore, the five parts must be three angles and two sides which means that the two triangles are similar. But, the two sides cannot be in corresponding order, or the triangles would be congruent either by ASA or SAS. So, the situation must be one of two possibilities as I have sketched below: (My sketches are not to scale.)

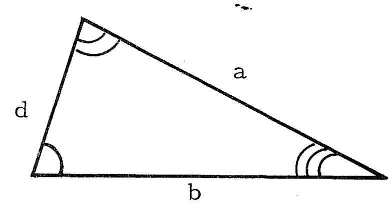


triangle 1

triangle 2



possibility 1



possibility 2

In both cases, by using relationships from similar triangles, it follows that $\frac{a}{b} = \frac{b}{c}$ or $b = ka$ and $c = kb = k^2a$ from possibility 2 and $\frac{a}{b} = \frac{b}{d}$ or $b = ka$ and $d = kb = k^2a$ from possibility 1.

So, the three sides of the triangle must be three consecutive members of a geometric series: a, ak, ak^2, where k is a proportionality constant and $k > 0$ and $k \neq 1$. If $k = 1$, the triangles would both be equilateral and thus congruent. Therefore, $k \neq 1$.

From my previous article on the Golden Section (Pentagon, Spring 1964) I worked out two problems on right triangles where the sides formed a geometric progression and the constants turned out to be $\sqrt{\tau}$ and $\sqrt{\frac{1}{\tau}}$. So, I knew of two more situations where the original problem could be solved. Then I began to consider various other values of k and I began to wonder what values of "k" will work. In other words, for what values of k will the numbers a, ak, and ak^2 be sides of a triangle. Once we know this, then another triangle with sides $\frac{a}{k}$, a, ak or ak, ak^2, ak^3 will have five parts congruent <u>but the triangles would not be congruent.</u>

In order for a, ak and ak^2 to be sides of a triangle, three statements must be true:

These are instances of the strict triangle inequality.

1. $a + ak > ak^2 \qquad (a + b > c)$

2. $a + ak^2 > ak \qquad (a + c > b)$

3. $ak + ak^2 > a \qquad (b + c > a)$

$$[a > 0, \quad k > 0, \quad k \neq 1]$$

For Case 1, consider $k > 1$

(a) $$k > 1 \rightarrow k^2 > k \rightarrow 1 + k^2 > k$$

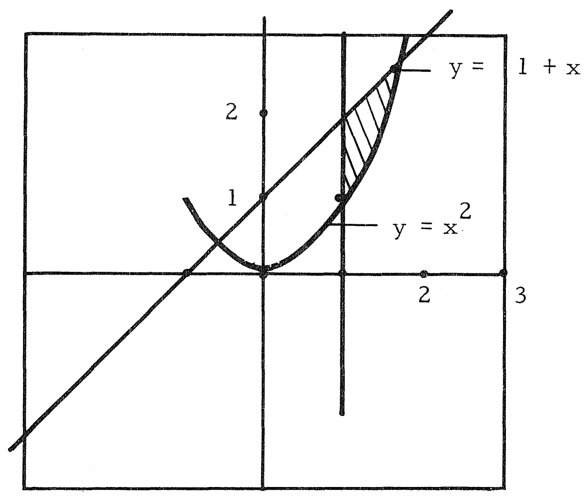
therefore, $a + ak^2 > ak$ (condition 2 above)

(b) $$k > 1 \rightarrow k + 1 > 1 \rightarrow k^2 + k > 1$$

therefore, $ak^2 + ak > a$ (condition 3 above)

(c) if $k > 1$ show $a + ak > ak^2$ (condition 1 above).

This part revolves around the problem of finding out when $1 + k > k^2$, or, graphically: For what $x > 1$ will $\underline{1 + x = y}$ be above $\underline{y = x^2}$?



Solving this problem produces the result that

$$k < \frac{1 + \sqrt{5}}{2} \quad \underline{\text{or}} \quad k < r :$$

So, if $1 < k < r$ then the numbers a, ak, ak^2 are the sides of the triangle that can be matched with $\frac{a}{k}$, a, ak or ak, ak^2, ak^3 to solve the original problem. (Incidentally: $1 < \sqrt{r} < r$. So this fits in here.)

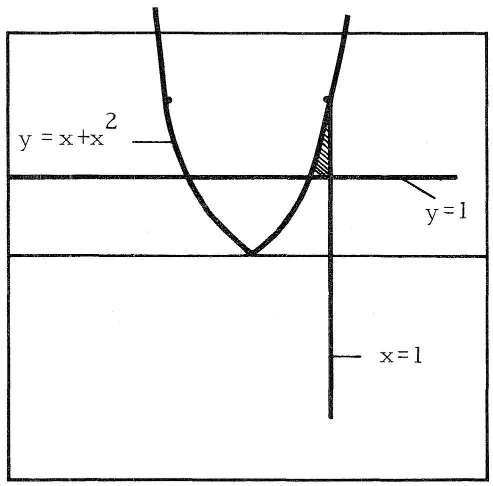
For Case 2, consider $k < 1$

(a) if $k < 1 \rightarrow k^2 < k \rightarrow k^2 < k + 1$ Therefore $ak^2 < ak + a$ (condition 1)

(b) if $k < 1 \rightarrow 1 + k > 1 \rightarrow a + ak^2 > ak$ (condition 2)

(c) Now, if $k < 1$ show $ak + ak^2 > a$. This is, essentially, finding what values of k make $k + k^2 > 1$.

Again, graphically, for what $x < 1$ will the parabola $y = x + x^2$ be above the line $y = 1$?



Solving this problem produces the result that $k > \frac{-1 + \sqrt{5}}{2}$. If you will follow this closely, $\frac{-1 + \sqrt{5}}{2}$ is the additive inverse of the conjugate of r. (i.e., $r = \frac{1 + \sqrt{5}}{2}$. Therefore, the conjugate of r is $\frac{1 - \sqrt{5}}{2}$ and its additive inverse is $\frac{-1 + \sqrt{5}}{2}$.) So, if $\frac{-1 + \sqrt{5}}{2} < k < 1$ the problem is again solved. (Again, $\frac{-1 + \sqrt{5}}{2} < \sqrt{\frac{1}{r}} < 1$, so my second problem fits here.)

Therefore, the complete solution can be summed up as follows, if k is a number such that $1 < k < \frac{1 + \sqrt{5}}{2} = r$ or $\frac{-1 + \sqrt{5}}{2} < k < 1$. Then the three sets of triangles with sides $\frac{a}{b}$, a, ak or a, ak, ak^2 or ak, ak^2, or ak^3 can be used to produce two triangles with five parts equal and the triangles themselves not congruent.

So, there are an infinite number of pairs of triangles that solve this problem and once again, r proves to be an interesting number and a key to the solution of interesting problems.

REFERENCES

1. Moise and Downs, Geometry, Addison-Wesley, p. 369.

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