CONCERNING LATTICE PATHS AND FIBONACCI NUMBERS

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R. E. Greenwood [1] has investigated plane lattice paths from (0, 0) to (n, n) and has found a relationship between the number of paths in a certain restricted subclass of such paths and the Fibonacci sequence. Considering such paths and using a method of enumeration different from that used by Greenwood, an unusual representation of Fibonacci's sequence is suggested.

The paths considered here are comprised of steps of three types: (i) horizontal from (x, y) to (x + 1, y); (ii) vertical from (x, y) to (x, y + 1); and (iii) diagonal from (x, y) to (x + 1, y + 1).

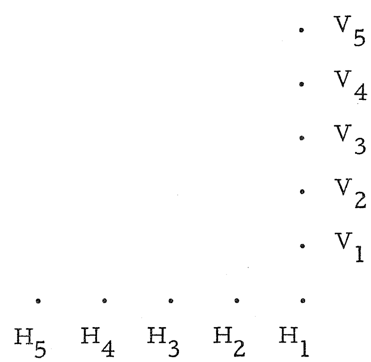

Figure 1

In the interest of simplicity of representation, we will here consider the paths from H_i to V_i, for each positive integer i. Note that the number of paths from H_i to V_i is the number of paths from (0, 0) to (i, i). However, instead of considering the total number of paths from H_i to V_i as was done by Greenwood, we will count only the number of paths from H_i to V_i which do not contain as subpaths any of the paths from H_j to V_j, for $j < i$. This number plus the number of paths from H_{i-1} to V_{i-1} is the total number of paths from H_i to V_i. The use of this counting device suggest the

Theorem:

Let

$$1_D = 1$$

$$2_D = \left[\frac{D-1}{2}\right], \text{ where } [\ ] \text{ denotes the greatest integer function}$$

$$3_D = 3_{D-1} + 2_{D-1}$$

$$4_D = 4_{D-2} + 3_{D-2}$$

$$\dots$$

$$(2n)_D = (2n)_{D-2} + (2n-1)_{D-2}$$

$$(2n+1)_D = (2n+1)_{D-1} + (2n)_{D-1}$$

$$\dots$$

with the restriction that $k_D = 0$ if $k > D$. For each positive integer D, let

$$f(D) = \sum_{k=1}^{D} k_D \ .$$

The sequence $\{f(D) \mid D = 1, 2, 3, \dots\}$ is the Fibonacci sequence.

The proof is direct and is therefore omitted.

The geometric interpretation of the numbers k_D and $f(D)$ mentioned in the theorem is interesting. However, before considering this interpretation it is necessary to define a section of a path. For this purpose we will now consider a path as the point set to which p belongs if and only if for some step $((x, y), (u, v))$ of the path, p belongs to the line interval whose end points are (x, y) and (u, v). A section of a path is a line interval which is a subset of the path and which is not a subset of any other line interval each of whose points is a point of the path.

The above mentioned geometric interpretation follows: By definition $f(1) = 1$. For each positive integer $D \geq 2$, let L_D denote the set of paths from H_D to V_D which do not contain as subpaths any of the paths from H_j to V_j, for $j < D$. $f(D)$ is the number of paths belonging to the set L_D. k_D is the number of paths in the subset X of L_D such that x belongs to X if and only if x contains as subsets exactly k diagonal sections.

Figure 2 portrays the five paths which belong to L_5. In Figure 2a appears the one path of L_5 which contains only one diagonal section ($1_5 = 1$). The two paths of L_5 which contain exactly two diagonal sections appear in Figure 2b ($2_5 = 2$). In Figure 2c the two paths of L_5 which contain exactly three diagonal sections are shown ($3_5 = 2$). It is noted that $4_5 = 5_5 = 0$.

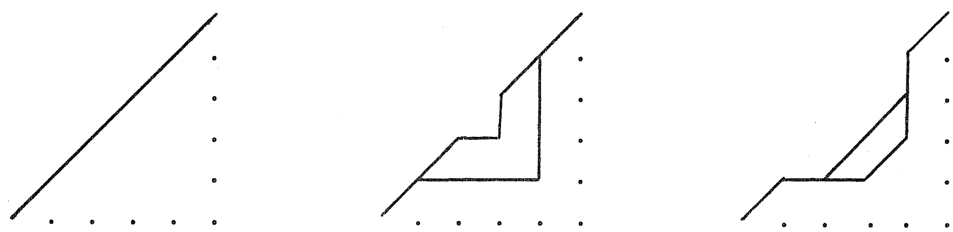

Fig. 2a $1_5 = 1$ | Fig. 2b $2_5 = 2$ | Fig. 2c $3_5 = 2$

$$f(5) = 1 + 2 + 2 + 0 + 0 = 5$$

Figure 2

REFERENCES

1. R. E. Greenwood, "Lattice Paths and Fibonacci Numbers," The Fibonacci Quarterly, Vol. 2, No. 1, pp. 13-14.

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