SUMMATION FORMULAE FOR MULTINOMIAL COEFFICIENTS

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1. INTRODUCTION

In [1] we have given some historical background to the multinomial coefficients and proved some of the basic summation formulae. More of the summation formulae can be found in [2]. In this paper we shall prove additional relations involving multinomial coefficients. Some of these can be considered as generalizations of corresponding formulae for binomial coefficients. We shall refer to [3] for these formulae.

2. FIRST SET OF FORMULAE

In order to simplify the notation used in [1], at least for the proof, we shall write

$$N!/\prod_{s=1}^{n} k_s! = \binom{N}{k_1, k_2, \ldots, k_n}, \quad \text{with,} \quad \sum_{s=1}^{n} k_s = N \;,$$

and, for, $k_1+k_2+\ldots+k_n = N+1$, we shall have the simplified notation

$$\binom{N}{k_1, k_2, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_n} = \left[N, (k_j-1), k_n\right] \;.$$

Under these conditions equation (6) of [1] can be written

$$\sum_{j=1}^{n} \left[N, (k_j-1), k_n\right] = \left[N+1, k_n\right] \;. \tag{1}$$

For $0 \leqq p \leqq n$, we can write (1) in the form

$$\sum_{j=1}^{p-1} \left[N, (k_j-1), k_p, k_n\right] + \left[N, k_p-1, k_n\right] + \sum_{j=p+1}^{n} \left[N, k_p, (k_j-1), k_n\right]$$
$$= \left[N+1, k_p, k_n\right] \;.$$

and similar relations for $N-1, N-2, \ldots, N-q, \ldots, N-k_p$, thus,

$$\sum_{j=1}^{p-1} \left[N-1, (k_j-1), k_p-1, k_n\right] + \left[N-1, k_p-2, k_n\right] + \sum_{j=p+1}^{n} \left[N-1, k_p-1, (k_j-1), k_n\right]$$
$$= \left[N, k_p-1, k_n\right]$$

. . . .

$$\sum_{j=1}^{p-1}\left[N-q, (k_j-1), k_p-q, k_n\right]+\left[N-q, k_p-q-1, k_n\right]+\sum_{j=p+1}^{n}\left[N-q, k_p-q, (k_j-1), k_n\right]$$

$$= \left[N-q+1, k_p-q, k_n\right]$$

. . . .

$$\sum_{j=1}^{p-1}\left[N-k_p, (k_j-1), 0, k_n\right]+\sum_{j=p+1}^{n}\left[N-k_p, 0, (k_j-1), k_n\right] = \left[N-k_p+1, 0, k_n\right] .$$

By adding the first q equations and simplifying we obtain

$$\sum_{\alpha=0}^{q}\left[\sum_{j=1}^{p-1}\left[N-\alpha, (k_j-1), k_p-\alpha, k_p\right]+\sum_{j=p+1}^{n}\left[N-\alpha, k_p-\alpha, (k_j-1), k_n\right]\right] =$$

$$= \left[N+1, k_p, k_n\right]-\left[N-q, k_p-q-1, k_n\right] ,$$

or, using the classical notation,

$$(2)\quad \sum_{\alpha=0}^{q}\left[\sum_{j=1}^{p-1}\binom{N-\alpha}{k_1, k_2, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_p-\alpha, \ldots, k_n} + \right.$$

$$\left.\sum_{j=p+1}^{n}\binom{N-\alpha}{k_1, k_2, \ldots, k_p-\alpha, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_n}\right] =$$

$$= \binom{N+1}{k_1, k_2, \ldots, k_p, \ldots, k_n} - \binom{N-q}{k_1, k_2, \ldots, k_p-q, \ldots, k_n} .$$

For $q = k_p$, we obtain

$$(3)\quad \sum_{\alpha=0}^{k_p}\left[\sum_{j=1}^{p-1}\binom{N-\alpha}{k_1, k_2, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_p-\alpha, \ldots, k_n} + \right.$$

$$\left.\sum_{j=p+1}^{n}\binom{N-\alpha}{k_1, k_2, \ldots, k_p-\alpha, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_n}\right] =$$

$$= \binom{N+1}{k_1, k_2, \ldots, k_p, \ldots, k_n} .$$

It will be noted that in both (2) and (3) the sum is independent of p, thus by summing on p we obtain

$$(4)\quad \sum_{p=1}^{n} \sum_{\alpha=0}^{k_p} \left[\sum_{j=1}^{p-1} \binom{N-\alpha}{k_1, k_2, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_p - \alpha, \ldots, k_n} + \sum_{j=p+1}^{n} \binom{N-\alpha}{k_1, k_2, \ldots, k_p - \alpha, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_n} \right] = n \binom{N+1}{k_1, k_2, \ldots, k_s, \ldots, k_n} .$$

For $n = 2$, (2) and (3) reduce to (3) and (4) of [3], p. 246.

3. SECOND SET OF FORMULAE

Consider the formulae leading to (2) and (3). If we multiply the first relation by $+1$, the second by -1, ..., the $(q+1)$-th relation by $(-1)^q$, etc., ... we obtain

$$(5)\quad \sum_{\alpha=0}^{q} \left[(-1)^{\alpha} \sum_{j=1}^{p-1} \binom{N-\alpha}{k_1, k_2, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_p - \alpha, \ldots, k_n} + \sum_{j=p+1}^{n} \binom{N-\alpha}{k_1, k_2, \ldots, k_p - \alpha, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_n} \right] =$$

$$2 \sum_{\alpha=1}^{q} (-1)^{\alpha} \binom{N-\alpha+1}{k_1, k_2, \ldots, k_p - \alpha, \ldots, k_n} + \binom{N-1}{k_1, k_2, \ldots, k_p, \ldots, k_n} + (-1)^{q+1} \binom{N-q}{k_1, k_2, \ldots, k_p - q - 1, \ldots, k_n} ,$$

and,

$$(6)\quad \sum_{\alpha=0}^{k_p} \left[(-1)^{\alpha} \sum_{j=1}^{p-1} \binom{N-\alpha}{k_1, k_2, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_p - \alpha, \ldots k_n} + \sum_{j=p+1}^{n} \binom{N-\alpha}{k_1, k_2, \ldots, k_p - \alpha, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_n} \right] =$$

$$2 \sum_{\alpha=1}^{k_p} \binom{N-\alpha+1}{k_1, k_2, \ldots, k_p - \alpha, \ldots, k_n} + \binom{N+1}{k_1, k_2, \ldots, k_p, \ldots, k_n} .$$

(5) and (6) for $n = 2$ reduce to (7) of [3], p. 247.

4. THIRD SET OF FORMULAE

Using the notation of section 2 we write for

$$\sum_{s=1}^{n} k_s = N+1, \quad k_p = 0 ,$$

$$\sum_{j=1}^{p-1} \left[N, (k_j-1), 0, k_n\right] + \sum_{j=p+1}^{n} \left[N, 0, (k_j-1), k_n\right] = \left[N+1, 0, k_n\right]$$

$$\sum_{j=1}^{p-1} \left[N+1, (k_j-1), 1, k_n\right] + \left[N+1, 0, k_n\right] + \sum_{j=p+1}^{n} \left[N+1, 1, (k_j-1), k_n\right] = \left[N+2, 1, k_n\right]$$

. . . .

$$\sum_{j=1}^{p-1} \left[N+q, (k_j-1), q, k_n\right] + \left[N+q, q-1, k_n\right] + \sum_{j=p+1}^{n} \left[N+q, q, (k_j-1), k_n\right] = \left[N+q+1, q, k_n\right]$$

. . . .

$$\sum_{j=1}^{p-1} \left[N+h, (k_j-1), h, k_n\right] + \left[N+h, h-1, k_n\right] + \sum_{j=p+1}^{b} \left[N+h, h, (k_j-1), k_n\right] = \left[N+h+1, h, k_n\right].$$

By adding and simplifying we obtain

$$\sum_{\alpha=0}^{q} \left[\sum_{j=1}^{p-1} \left[N+\alpha, (k_j-1), \alpha, k_n\right] + \sum_{j=p+1}^{n} \left[N+\alpha, \alpha, (k_j-1), k_n\right] \right] = \left[N+q, q, k_n\right] ,$$

or, using the classical notation

$$(7) \quad \sum_{\alpha=0}^{q} \left[\sum_{j=1}^{p-1} \binom{N+\alpha}{k_1, k_2, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, \alpha, \ldots, k_n} + \sum_{j=p+1}^{n} \binom{N+\alpha}{k_1, k_2, \ldots, \alpha, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_n} \right] =$$

$$\binom{N+q+1}{k_1, k_2, \ldots, k_{p-1}, q+1, k_{p+1}, \ldots, k_n} ,$$

and similarly

$$(8)\quad \sum_{\alpha=q}^{h}\left[\sum_{j=1}^{p-1}\binom{N+\alpha}{k_1, k_2, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, \alpha, \ldots, k_n} + \sum_{j=p+1}^{n}\binom{N+\alpha}{k_1, k_2, \ldots, \alpha, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_n}\right] = \binom{n+h+1}{k_1, k_2, \ldots, k_{p-1}, h+1, k_{p+1}, \ldots, k_n} - \binom{n+q-1}{k_1, k_2, \ldots, k_{p-1}, q-1, k_{p+1}, \ldots, k_n} .$$

For $n = 2$ (8) reduces to (11) of [3] p. 248.

5. FOURTH SET OF FORMULAE

(8) of [1] can be simplified in writing by introducing the notation

$$(9)\quad \sum_{j_1=0}^{k_1}\sum_{j_2=0}^{k_2}\cdots\sum_{j_{n-1}=0}^{k_{n-1}} = \left(\prod_{s=1}^{n-1}\cdot\sum_{j_s=0}^{k_s}\right),$$

where Π operates on the operator Σ. Under these conditions (8) of [1] can be written for,

$$\sum_{s=1}^{n} k_s = p+q, \quad \sum_{s=1}^{n} j_s = p \quad ,$$

$$(10)\quad \left(\prod_{s=1}^{n-1}\cdot\sum_{j_s=0}^{k_s}\right)\binom{p}{j_1, j_2, \ldots, j_n}\binom{q}{k_1-j_1, k_2-j_2, \ldots, k_n-j_n} = \binom{p+q}{k_1, k_2, \ldots, k_n} .$$

Let us substitute $p+r$ for p in (10), we obtain for $\binom{p}{j_1, j_2, \ldots, j_n}\cdot$

$$\binom{p+r}{j_1, j_2, \ldots, j_n} = \left(\prod_{t=1}^{n-1}\cdot\sum_{h_t=0}^{j_t}\right)\binom{p}{j_1-h_1, j_2-h_2, \ldots, j_n-h_n}\binom{r}{h_1, h_2, \ldots, h_n} ,$$

with

$$\sum_{i=1}^{n} h_i = r, \quad \sum_{i=1}^{n} j_i = p+r, \quad \sum_{i=1}^{n} k_i = p+q+r \quad ,$$

so that substituting into (10) we obtain

$$(11) \quad \left(\prod_{s=1}^{n-1} \cdot \sum_{j_s=0}^{k_s}\right)\left(\prod_{t=1}^{n-1} \cdot \sum_{h_t=0}^{j_t}\right)\binom{p}{j_1-k_1, j_2-k_2, \ldots, j_n-k_n}\binom{q}{k_1-j_1, \ldots, k_n-j_n} \cdot$$

$$\cdot \binom{r}{h_1, h_2, \ldots, h_n} = \binom{p+q+r}{k_1, k_2, \ldots, k_n} \quad .$$

More generally as can be proved by induction we can write

$$(12) \quad \prod_{j=1}^{m-1} \cdot \left(\prod_{i=1}^{n-1} \cdot \sum_{k_{j+1,i}=0}^{k_{i,j}}\right) \prod_{j=1}^{m-1} \binom{q_j}{k_{j,1}-k_{j+1,1}, k_{j,2}-k_{j+1,2}, \ldots, k_{j,n}-k_{j+1,n}}.$$

$$\cdot \binom{q_m}{k_{m,1}, k_{m,2}, \ldots, k_{m,n}} = \binom{q_1+q_2+\ldots+q_m}{k_{11}, k_{12}, \ldots, k_{1n}} \quad ,$$

where,

$$\sum_{t=1}^{n} (k_{j,t}-k_{j+1,t}) = q_j, \quad \text{for,} \quad j=1, 2, \ldots, n \quad .$$

REFERENCES

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3. E. Netto, Lehrbuch der Combinatorik, reprint of second Ed., Chelsea, N. Y., 1958.

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