TIME GENERATED COMPOSITIONS YIELD FIBONACCI NUMBERS

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1. INTRODUCTION

Imagine a particle the number of whose collisions with other particles during the t^{th} time interval is given by $\phi(t)$. Assume that this particle possesses a property, p, which it can transmit by collision to every particle with which it collides. Further suppose that every particle that has received property p by collision can also transmit it by collision. Assume that for an indefinite period of time every particle possessing property, p, collides only with those particles not possessing this property. The number of new particles to which property, p, has been imparted is given by the following model.

2. THE MODEL

Let Δ_i be the number of collisions with new particles in the time interval $i < t \leq i + 1$ by particles possessing property, p, at $t = i$. The new particles do not start their private times until the end of the time interval of their initial collision.

(1)
$$\begin{aligned}
\Delta_0 &= 1 \\
\Delta_1 &= \phi(1) \\
\Delta_2 &= \phi(2) + \phi^2(1) \\
\Delta_3 &= \phi(3) + 2\phi(2)\phi(1) + \phi^3(1) \\
\Delta_4 &= \phi(4) + [2\phi(3)\phi(1) + \phi^2(2)] + 3\phi(2)\phi^2(1) + \phi^4(1) \\
&\cdots \\
\Delta_i &= F(h_i, \phi)
\end{aligned}$$

The model is obtained as follows:

Up to $t = 1$, Δ_0 generates the increment Δ_1, whose magnitude is $\phi(1)$, the number of particles with which Δ_0 collided in the first time interval.

At the time $t = 2$, Δ_0 has collided with $\phi(2)$ more new particles during the second time interval and Δ_1 has collided with $\phi(1)$ new particles, since its collisions are subject to the phase rule constraint of its own private time. Therefore when $t = 2$ in public time, $\Delta_2 = \phi(2) + \phi(1)\phi(1) = \phi(2) + \phi^2(1)$.

When $t = 3$, Δ_0 has collided with $\phi(3)$ more new particles during the third time interval for it is in phase 3 of its private time, each particle of $\Delta_1 = \phi(1)$ has collided with $\phi(2)$ more new particles, producing $\phi(1)\phi(2)$ new particles altogether, because Δ_1 is in the second phase of its private time. Each particle of Δ_2 collides with $\phi(1)$ new particles since it is in the first phase of its private time, thus producing

$$\Delta_2\phi(1) = (\phi(2) + \phi^2(1))\phi(1) = \phi(2)\phi(1) + \phi^3(1)$$

particles. Therefore when $t = 3$, we have

$$\begin{aligned}\Delta_3 &= [\phi(3)] + [\phi(1)\phi(2)] + [\phi(2)\phi(1) + \phi^3(1)] \\ &= \phi(3) + 2\phi(2)\phi(1) + \phi^3(1)\quad.\end{aligned}$$

Now if we substitute $\phi(t) = t$ into the model display (1), we obtain

(2)
$$\begin{aligned}\Delta_0 &= 1 \\ \Delta_1 &= 1 \\ \Delta_2 &= 2 + 1^2 = 3 \\ \Delta_3 &= 3 + 2^2 \cdot 1 + 1^3 = 8 \\ \Delta_4 &= 4 + 2 \cdot 3 \cdot 1 + 2^2 + 3 \cdot 2 \cdot 1^2 + 1^4 = 21\end{aligned}$$

Neglecting Δ_0, one observes that the numbers 1, 3, 8, 21, 55, ..., $U_{n+2} = 3U_{n+1} - U_n$ are the alternate terms of the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots, F_{n+2} = F_{n+1} + F_n$$

so that the sequence of cumulative sums (including Δ_0) is

$$1,\ 1 + 1 = 2,\ 1 + 1 + 3 = 5,\ 1 + 1 + 3 + 8 = 13,\ \ldots,$$

$U_{n+2} = 3U_{n+1} - U_n$ which is the other set of alternate Fibonacci numbers. The proof of these statements will follow as a special case of the theorem in the following section.

3. ANOTHER SPECIAL MODEL

If we assume that the time generator is $\phi(t) = kt$ (k a positive integer), the same model display (1) yields

(3)
$$\begin{aligned}\Delta_0 &= 1\\ \Delta_1 &= k\\ \Delta_2 &= k^2 + 2k\\ \Delta_3 &= k^3 + 4k^2 + 3k\\ \Delta_4 &= k^4 + 6k^3 + 10k^2 + 4k\\ &\ldots\ldots\ldots\ldots\\ \Delta_i &= P_i(k)\end{aligned}$$

Note: The coefficient of k^m in the polynomial $P_n(k)$ is the number of distinct compositions of integer n in m positive integers. The coefficients are <u>also</u> the alternate rising diagonals of Pascal's arithmetic triangle upward from left to right.

We now prove the following theorem.

Theorem: If $\phi(t) = kt$, then model display (3) has as its nth row a polynomial $P_n(k)$ satisfying the recursion relation:

$$P_{n+2}(k) = (k + 2)P_{n+1}(k) - P_n(k) \quad ,$$

where $P_1(k) = k$ and $P_2(k) = k^2 + 2k$.

4. PROOF OF THE THEOREM

Let $T_n(k)$ be the total number of particles possessing property, p, at time $t = n$. Clearly $T_{n+1}(k) = T_n(k) + \Delta_{n+1}$, while collectively the $T_n(k)$ particles collide with Δ_{n+1} new particles during the next time interval, each particle collides with k more new particles than during the previous time interval so that

(4) $$\Delta_{n+2} = k(T_n(k) + \Delta_{n+1}) + \Delta_{n+1} = kT_{n+1}(k) + \Delta_{n+1} \quad .$$

Thus, since $\Delta_{n+1} = T_{n+1}(k) - T_n(k)$ equation (4) yields

(5) $$T_{n+2}(k) = (k+2)\, T_{n+1}(k) - T_n(k) \quad .$$

But, since $\Delta_{n+1} = T_{n+1}(k) - T_n(k)$ is the difference of two solutions of (5), it is also a solution of (5). Now, $\Delta_1 = k = P_1(k)$ and $\Delta_2 = k^2 + 2k = P_2(k)$ and the proof is complete. If $k = 1$, then (5) becomes

(6) $$U_{n+2} = 3U_{n+1} - U_n$$

If $U_1 = P_1(1) = 1$, and $U_2 = P_2(1) = 3$, then the numbers generated are the alternate Fibonacci numbers promised after (2), while

$$U_0 = T_0(1) = \Delta_0 = 1, \text{ and } U_1 = T_1(1) = \Delta_0 + \Delta_1 = 1 + 1 = 2 \quad ,$$

recursion relation (6) yields the other set of alternate Fibonacci numbers as the sequence of cumulative sums, the total particle count.

5. CONCLUDING REMARKS

One is directed to advanced problem H-50 December 1964, Fibonacci Quarterly, for the partitioning interpretation of the integer n of the model for $\phi(t) = kt$.

Suppose one defines two sets of Morgan-Voyce polynomials

$$b_0(x) = 1,\ b_1(x) = 1 + x;\ B_0(x) = 1,\ B_1(x) = 2 + x\quad,$$

both sets satisfying

$$P_{n+2}(x) = (x + 2)\,P_{n+1}(x) - P_n(x),\quad n \geq 0\quad. \tag{7}$$

It is easy to establish that

$$P_n(k) = \Delta_n = k\,B_{n-1}(k)$$

$$T_n(k) = \Delta_0 + \Delta_1 + \dots + \Delta_n = b_n(k)\quad.$$

Thus for $k = 1$, we again find $B_{n-1}(1) = F_{2n}$ and $b_n(1) = F_{2n+1}$. See corrected problem B-26 with solution by Douglas Lind in the Elementary Problem Section of this issue, where the binomial coefficient relation mentioned in the note of Section 3 is shown. A future paper by Prof. M. N. S. Swamy dealing extensively with Morgan-Voyce polynomials will appear in an early issue of the Fibonacci Quarterly.

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Additional references to work along the lines of generated compositions — some of which yield numbers with Fibonacci properties — will be found in the references at the end of this paper. (See note, page 94)

REFERENCES

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