Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within two months of publication.

B-70 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Denote $x^a$ by ex(a). Show that the following expression, containing $n$ integrals,

$$\int_0^1 \text{ex} \left( \int_0^1 \text{ex} \left( \int_0^1 \text{ex} \left( \int_0^1 x \, dx \right) \, dx \right) \, dx \right) \, dx$$

equals $F_{n+1}/F_{n+2}$, where $F_n$ is the $n$-th Fibonacci number.

B-71 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Find $a^{-2} + a^{-3} + a^{-4} + \ldots$, where $a = (1 + 5)/2$.

B-72 Proposed by J. A. H. Hunter, Toronto, Canada

Each distinct letter in this simple alphabetic stands for a particular and different digit. We all know how rabbits link up with the Fibonacci series, so now evaluate our RABBITS.

R A B B I T S

B E A R

R A B B I T S

A S

A S E R I E S

B-73 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Prove that

$$\sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n}{k} \binom{k+r-j-1}{j} = 1 + \sum_{m=0}^{2n+r-2} \sum_{p=0}^{m} \binom{m-p-1}{p} ,$$

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The Fibonacci polynomial $f_n(x)$ is defined by $f_1 = 1$, $f_2 = x$, and $f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$ for $n > 2$. Show the following:

(a) $x \sum_{r=1}^{n} f_r(x) = f_{n+1} + f_{n-1}$.

(b) $f_{m+n+1} = f_{m+1} f_{n+1} + f_{m} f_{n}$.

(c) $f_n(x) = \sum_{j=0}^{[n-1]/2} \binom{n-1}{j} x^{n-2j-1}$,

where $[k]$ is the greatest integer not exceeding $k$. Hence show that the $n$-th Fibonacci number

$$F_n = \sum_{j=0}^{[n-1]/2} \binom{n-1}{j}.$$

B-75 Proposed by M. N. S. Swamy, University of Saskatchewan, Regina, Canada

Let $f_n(x)$ be as defined in B-74. Show that the derivative

$$f'_n(x) = \sum_{r=1}^{n-1} f_r(x) f_{n-r}(x) \text{ for } n > 1.$$

SOLUTIONS

ONE, TWO, THREE — OUT

B-58 Proposed by Sidney Kravitz, Dover, New Jersey

Show that no Fibonacci number other than 1, 2, or 3 is equal to a Lucas number.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

Since $L_k = F_{k-1} + F_{k+1}$, the assertion is equivalent to

(1) $F_n = F_{k-1} + F_{k+1}$.
If \( k \geq 3 \), the \( n > k+1 \) and (1) is clearly impossible since

\[
F_{k-1} + F_{k+1} < F_k + F_{k+1} = F_{k+2} < F_n.
\]

Impossibility for \( k \geq 3 \) implies impossibility for \( k \leq -3 \) since only signs are different. For \(-3 < k < 3\) we find \( F_{-2} = L_{-1} = 1 \), \( F_3 = l_0 = 2 \), \( F_1 = L_1 = 1 \), and \( F_4 = L_2 = 3 \), corresponding to \( k = -1 \), 0, 1, and 2 respectively. Hence these are the only solutions.

(The crux of this problem is solved in the discussion of equation (12) in Carlitz', "A Note on Fibonacci Numbers," this Quarterly 1 (1964) No. 2 pp. 15-28).

Also solved by J. L. Brown, Jr.; Gary C. McDonald; C. B. A. Peck; and the proposer.

B-59 Proposed by Brother U. Alfred, St. Mary’s College, California

Show that the volume of a truncated right circular cone of slant height \( F_n \) with \( F_{n-1} \) and \( F_{n+1} \) the diameters of the bases is

\[
\sqrt{3} \pi (F_{n+1}^3 - F_{n-1}^3)/24.
\]

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

It is well-known that if \( h \) is the height of the frustrum of a right circular cone, \( s \) the slant height, and \( r_1 \) and \( r_2 \) the radii of the bases, then the volume \( V \) is

\[
V = \frac{(\pi h / 3) (r_1^2 + r_1 r_2 + r_2^2)}{(r_1^2 + r_2^2 - s^2)}.
\]

For this problem \( r_1 = F_{n-1}/2 \), \( r_2 = F_{n+1}/2 \) and \( s = F_n \), so that

\[
V = \frac{\pi}{3} \sqrt{F_n^2 - (F_{n+1}^2 - F_{n-1}^2)/4} (F_{n-1}^2 + F_{n+1}^2 + 2F_{n+1}/F_{n+1})/4
\]

\[
= \sqrt{3} \pi F_n (F_{n-1}^2 + F_{n+1}^2 + 2F_{n+1})/24
\]
= $\sqrt{3}\pi (F_{n+1} - F_{n-1})(F_{n-1} + F_{n-1}F_{n+1} + F_{n+1}^2)/24$

= $\sqrt{3}\pi (F_{n+1}^3 - F_{n-1}^3)/24$.

We remark that the area $A$ of the curved surface of the frustum is

$$A = \pi F_n (F_{n+1} + F_{n-1})/2 = (\pi/2)F_n L_n.$$

Also solved by Carole Bania, Gary C. McDonald, Kenneth E. Newcomer, C. B. A. Peck, M. N. S. Swamy, Howard L. Walton, John Wessner, Charles Ziegenfus, and the proposer. McDonald also added the formula for the curved surface.

B-60 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California

Show that $L_{2n}L_{2n+2} + 5F_{2n+1}^2 = 1$, where $F_n$ and $L_n$ are the $n$-th Fibonacci number and Lucas number, respectively.


Using my second answer to B-22 (Vol. 2, No. 1, p. 78),

$$L_{2(n+1)}L_{2n} = 5F_{2(n+1)+n}^2 + L_{2(n+1)-n}^2$$

$$= 5F_{2n+1}^2 + L_1^2$$

$$= 5F_{2n+1}^2 + 1$$

Thus

$$L_{2n+2}L_{2n} - 5F_{2n+1}^2 = 1.$$


MODULO THREE

B-61 Proposed by J. A. H. Hunter, Toronto, Ontario

Define a sequence $U_1$, $U_2$, ... by $U_1 = 3$ and

$$U_n = U_{n-1} + n^2 + n + 1$$

for $n > 1$.

Prove that $U_n \equiv 0 \pmod{n}$ if $n \neq 0 \pmod{3}$.

Solution by John Wessner, Melbourne, Florida
An alternative representation for $U_n$ is

$$U_n = \sum_{k=1}^{n} (k^2 + k + 1).$$

Upon expanding the individual sums involved we obtain

$$U_n = \left[\frac{n(2n+1)(n+1)}{6}\right] + \left[\frac{n(n+1)}{2}\right] + n = \frac{n}{3}[n+2](n+1)+3].$$

Hence, $U_n \equiv 0 \pmod{n}$ if and only if $(n+1)(n+2) \equiv 0 \pmod{3}$. This condition obtains if and only if $n \neq 0 \pmod{3}$.

Also solved by Robert J. Hursey, Jr., Douglas Lind, Gary C. McDonald, Robert McGee, C. B. A. Peck, Charles R. Wall, David Zeitlin, and the proposer.

**UNIQUE SUM OF SQUARES**

**B-62** Proposed by Brother U. Alfred, St. Mary's College, California

Prove that a Fibonacci number with odd subscript cannot be represented as the sum of squares of two Fibonacci numbers in more than one way.

Solution by J. L. Brown, Jr., Pennsylvania State University, State College, Pa.

From the identity $F_{2n+1}^2 = F_n^2 + F_{n+1}^2$, \( n \geq 1 \) it follows that $F_{2n+1}^2 < (F_n^2 + F_{n+1}^2)^{n+2}$. Therefore, any representation, $F_{2n+1}^2 = F_k^2 + F_m^2$ \( k, m \leq n+1 \). Then $k \geq n$ (otherwise $F_k^2 + F_m^2 < F_n^2 + F_{n+1}^2 = F_{2n+1}^2$ for $k > 2$).

Also solved by Douglas Lind, Joseph A. Orjechouski and Robert McGee (jointly), C. B. A. Peck, and the proposer.

**AN ISOSCELES TRIANGLE**

**B-63** An old problem whose source is unknown, suggested by Sidney Kravitz, Dover, New Jersey.

In $\triangle ABC$ let sides $AB$ and $AC$ be equal. Let there be a point $D$ on side $AB$ such that $AD = CD = BC$. Show that

$$2\cos \theta A = AB/BC = (1 + \sqrt{5})/2 ,$$

the golden mean.

Solution by John Wessner, Melbourne, Florida

By inspection of the figure and the law of cosines

$$AD^2 = CD^2 + AC^2 - 2CD \cdot AC \cos \theta A.$$
Since $AD = CD = BC$ and $AB = AC$, it follows immediately that 

$$2 \cos \frac{\angle A}{2} = \frac{AC}{CD} = \frac{AB}{BC}.$$ 

The second result comes from the fact that 

$$\frac{\angle B}{2} = \frac{\angle BDC}{2} = \frac{\angle A}{2} + \frac{\angle DCA}{2} = 2 \times \frac{\angle A}{2}$$ 

and hence 

$$\angle A = 36^\circ \text{ and } 2 \cos A = \frac{1 + \sqrt{5}}{2}.$$ 

(See N. N. Vorob'ev: The Fibonacci Numbers (New York, 1961) p. 56.) 


Continued from page 234.


ASSOCIATION PUBLISHES BOOKLET

Brother U. Alfred has just completed a new booklet entitled: Introduction to Fibonacci Discovery. This booklet for teachers, researchers, and bright students can be secured for $1.50 each or 4 copies for $5.00 from Brother U. Alfred, St. Mary's College, Calif.