AN ELEMENTARY METHOD OF SUMMATION

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The purpose of this note is to present an elementary method for summing the first n terms of a sequence which satisfies a given homogeneous linear recursion relation. The method is, in fact, a simple extension of that normally used for summing a geometric progression, which we first recall.

Let:

$$S = a + ar + ar^2 + \ldots + ar^n$$

Then:

$$-rS = \quad - ar - ar^2 - \ldots - ar^n - ar^{n+1}$$

Therefore:

$$S(1 - r) = a - ar^{n+1}$$

and if $r \neq 1$,

$$S = \frac{a - ar^{n+1}}{1 - r} .$$

We now turn to the general case. If for every positive integer j, G_j satisfies

$$G_{j+k} + \sum_{i=1}^{k} c_i G_{j+k-i} = 0, \tag{1}$$

where the c_i are fixed quantities, we write, as above

$$\begin{array}{llll|llll|l}
S = & G_1 + G_2 + G_3 + \ldots & + & & G_{k+1} & + G_{k+2} & + \ldots & + G_n & \\
c_1 S = & c_1 G_1 + c_1 G_2 + \ldots & + & & c_1 G_k & + c_1 G_{k+1} & + \ldots & + c_1 G_{n-1} & + c_1 G_n \\
c_2 S = & \quad c_2 G_1 + \ldots & + & & c_2 G_{k-1} & + c_2 G_k & + \ldots & + c_2 G_{n-2} & + c_2 G_{n-1} + c_2 G_n \\
\ldots & & & & & & & & \\
c_k S = & & & & c_k G_1 & + c_k G_2 & + \ldots & + c_k G_{n-k} & + \ldots + c_k G_n
\end{array}$$

Since, adding vertically and using (1), the sum of the terms inside the dotted lines is zero, we see:

$$S(1+c_1+\ldots+c_k) = G_1(1+c_1+\ldots+c_{k-1}) + G_2(1+c_1+\ldots+c_{k-2}) + \ldots + G_k$$

$$+ G_n(c_1+c_2+\ldots+c_k) + G_{n-1}(c_2+\ldots+c_k) + \ldots + c_k G_{n-k+1}.$$

If $1 + c_1 + c_2 + \ldots + c_{k-1} \neq 0$, we can solve for S.

The same method can be used to find

$$\sum_{i=1}^{n} i^t G_i \text{ , for a given } t,$$

if the G_i satisfy (1). To facilitate the presentation, we collect some terminology and facts.

Let E be the operator with the property that

$$EG_i = G_{i+1} \text{ .}$$

To say that G_j satisfies (1) is equivalent to the statement that the operator

$$\phi(E) = E^k + \sum_{i=1}^{k} c_i E^{k-i}$$

when applied to any G_j, yields zero (E^0 being the identity operator). The associated polynomial

$$\phi(x) = x^k + \sum_{i=1}^{k} c_i x^{k-i}$$

is called the characteristic polynomial.* The special role of the number one in our generalization is now easily stated, for

$$1 + c_1 + \ldots + c_k \neq 0$$

if and only if unity is not a root of the characteristic polynomial.

*$\phi(x)$ is unique if we assume $\psi(E)\,G_j = 0$ for all positive j implies the degree of $\psi(x) \geq k$.

It is known ([2] , pp. 548-552) that if $\phi(E)G_j = 0$, then $B_j = j^{t-1} G_j$ satisfies

$$[\phi(E)]^t B_j = 0, \text{ for } t \geq 1.$$

If $\phi(1) \neq 0$ then $\psi(1) \neq 0$, where $\psi(x) = [\phi(x)]^t$, and the method just described can be used to find

$$T = \sum_{j=1}^{n} B_j = \sum_{j=1}^{n} j^{t-1} G_j .$$

Writing

$$\psi(x) = x^{kt} + \sum_{i=1}^{kt} d_i x^{kt-i},$$

we find:

$$p_0 T = \sum_{j=1}^{kt-j} p_j B_j + \sum_{j=0}^{kt-1} r_j B_{n-j}$$

where

$$p_j = 1 + \sum_{i=1}^{kt-j} d_i \text{ and } r_j = \sum_{i=j+1}^{kt} d_i .$$

Since $\phi(E)G_j = 0$ and $B_j = j^t G_j$, one can easily obtain T in terms of $G_1, \ldots, G_{k-1}; G_{n-k+2}, \ldots, G_n$.

The assumption that unity not be a root of the characteristic polynomial has been critical to our discussion so far. We now assume $\{G_j\}$ satisfies

$$\chi(E)\, G_j = 0$$

where $\chi(E)$ is a polynomial with $\chi(1) = 0$. Factoring out all the factors $x - 1$ in $\chi(x)$, we obtain

$$\chi(x) = (x - 1)^a \phi(x), \text{ where } \phi(1) \neq 0.$$

Letting $C_j = \phi(E)G_j$, we note:

$$\phi(1)S = \phi\frac{(1)}{1}\sum_{j=1}^{n} G_j = \sum_{j=1}^{k} G_j q_j + \sum_{j=1}^{n-k} C_j + \sum_{j=0}^{k-1} G_{n-j} s_j ,$$

where

$$\chi(x) = x^k + \sum_{i=1}^{k} c_i x^{k-i},$$

$$q_j = 1 + \sum_{i=1}^{k-j} c_i \quad \text{and} \quad s_j = \sum_{i=j+1}^{k} c_i \quad .$$

However, it is known ([2], pp. 548-552) that if $(E - 1)^a C_j = 0$, then C_j is a polynomial of degree $\leq a - 1$. If we assume the formulas for

$$\sum_{j=1}^{n} j^p$$

are known, for j a positive integer, the only problem remaining is that of determining the polynomial $C_j = d_0 + d_1 j + \ldots + d_{a-1} j^{a-1}$. It is easy to show that the difference operator E-1 when applied to a polynomial of degree r yields a polynomial of degree r - 1. Therefore $(E - 1)^j C_1$ involves only $d_{a-1}, d_{a-2} \ldots, d_j$ and the system of linear equations on the d_i obtained by computing $(E - 1)^j C_1$, j = 0, 1, 2, ..., a - 1 can clearly be solved for the d_i.

The above is a generalization of the technique used by Erbacher and Fuchs to solve problem H-17. [4]

Example: Assume that for each positive integer j, G_j satisfies $\chi(E)G_j = 0$, where $\chi(x) = (x - 1)^3 (x^3 - 3x^2 + 4x + 2) = (x-1)^3 \phi(x)$. If $G_1 = G_2 = G_3 = G_4 = G_5 = 0$, $G_6 = 1$, then $C_1 = \phi(E)G_1 = 0$, $C_2 = \phi(E) G_2 = 0$, $C_3 = \phi(E)\ G_3 = 1$. With $C_j = d_0 + d_1 j + d_2 j^2$, we find $(E - 1)^2 C_1 = 2d_2 = 1$, $(E - 1)\ C_1 = d_1 + 3d_2 = 0$ and $C_1 = d_0 + d_1 + d_2$. Hence $C_j = 1 - (3/2)\ j + j^2/2$ and

$$\phi(1)\,S = 4S = 2G_1 - 2G_2 + G_3 + \sum_{j=1}^{n-3} (1 - (3j)/2 + j^2/2) + 3G_n + 6G_{n-1} + 2G_{n-2}$$

$$= \sum_{j=1}^{n-3} (1 - (3j)/2 + j^2/2) + 3G_n + 6G_{n-1} + 2G_{n-2} \ .$$

In conclusion, we have seen how the elementary method used to sum a geometric progression can be generalized to find the sum of the first n terms of a sequence which satisfies a linear homogeneous recursion relation. It may be worth stating that this method is applicable to a sequence whose terms are products of corresponding terms of sequences each of which satisfy a linear homogeneous recursion relation (see [1] pp. 42-45 for a special case).

We propose as a problem for the reader: Find in closed form the sum of the first n terms of the sequence $\{w_n\}$:

$$1, 2, 10, 36, 145, \ . \ . \ .$$

where $w_n = F_n G_n$ with $F_{n+2} = F_{n+1} + F_n$ $(F_1 = F_2 = 1)$ and $G_{n+2} = 2G_{n+1} + G_n$ $(G_1 = 1,\ G_2 = 2)$.

REFERENCES

1. Dov Jarden, Recurring Sequences, Jerusalem, 1958.
2. C. Jordan, "Calculus of Finite Differences," Chelsea, New York, Ed. 1950.
3. James A. Jeske, "Linear Recurrence Relations, Part I," The Fibonacci Quarterly, Vol. 1, No. 2, pp. 69-74.
4. Problem H-17, The Fibonacci Quarterly, Proposed in Vol. 1, No. 2, 1963, p. 55 and solved by Joseph Erbacher and John Allen Fuchs in Vol. 2, No. 1, 1964, p. 51.

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