The purpose of this note is to present an elementary method for summing the first $n$ terms of a sequence which satisfies a given homogeneous linear recursion relation. The method is, in fact, a simple extension of that normally used for summing a geometric progression, which we first recall.

Let:

$$S = a + ar + ar^2 + \ldots + ar^n$$

Then:

$$-rS = -ar - ar^2 - \ldots - ar^n - ar^{n+1}$$

Therefore:

$$S(1 - r) = a - ar^{n+1}$$

and if $r \neq 1$,

$$S = \frac{a - ar^{n+1}}{1 - r}.$$ 

We now turn to the general case. If for every positive integer $j$, $G_j$ satisfies

$$G_{j+k} + \sum_{i=1}^{k} c_i G_{j+k-i} = 0,$$

where the $c_i$ are fixed quantities, we write, as above

$$S = G_1 + G_2 + G_3 + \ldots + G_{k+1} + G_{k+2} + \ldots + G_n$$

$$c_1S = c_1G_1 + c_1G_2 + \ldots + c_1G_k + c_1G_{k+1} + \ldots + c_1G_{n-1} + c_1G_n$$

$$c_2S = c_2G_1 + \ldots + c_2G_{k-1} + c_2G_k + \ldots + c_2G_{n-2} + c_2G_{n-1} + c_2G_n$$

$$\ldots$$

$$c_kS = c_kG_1 + \ldots + c_kG_{n-k} + \ldots + c_kG_n$$
Since, adding vertically and using (1), the sum of the terms inside the dotted lines is zero, we see:

\[ S(l + c_1 + \ldots + c_k) = G_1(l + c_1 + \ldots + c_{k-1}) + G_2(l + c_1 + \ldots + c_{k-2}) + \ldots + G_k + G_n(c_1 + c_2 + \ldots + c_k) + \ldots + c_k G_{n-k+1}. \]

If \( 1 + c_1 + c_2 + \ldots + c_{k-1} \neq 0 \), we can solve for \( S \).

The same method can be used to find

\[ \sum_{i=1}^{n} i^t G_i, \text{ for a given } t, \]

if the \( G_i \) satisfy (1). To facilitate the presentation, we collect some terminology and facts.

Let \( E \) be the operator with the property that

\[ EG_i = G_{i+1}. \]

To say that \( G_j \) satisfies (1) is equivalent to the statement that the operator

\[ \phi(E) = E^k + \sum_{i=1}^{k} c_i E^{k-i} \]

when applied to any \( G_j \), yields zero (\( E^0 \) being the identity operator).

The associated polynomial

\[ \phi(x) = x^k + \sum_{i=1}^{k} c_i x^{k-i} \]

is called the characteristic polynomial. The special role of the number one in our generalization is now easily stated, for

\[ 1 + c_1 + \ldots + c_k \neq 0 \]

if and only if unity is not a root of the characteristic polynomial.

\[ \phi(x) \] is unique if we assume \( \psi(E) G_j = 0 \) for all positive \( j \) implies the degree of \( \psi(x) \geq k. \)
It is known ([2], pp. 548-552) that if \( \phi(E)G_j = 0 \), then \( B_j = j^{t-1} G_j \) satisfies

\[
[\phi(E)]^t B_j = 0, \text{ for } t \geq 1.
\]

If \( \phi(1) \neq 0 \), then \( \psi(1) \neq 0 \), where \( \psi(x) = [\phi(x)]^t \), and the method just described can be used to find

\[
T = \sum_{j=1}^{n} B_j = \sum_{j=1}^{n} j^{t-1} G_j.
\]

Writing

\[
\psi(x) = x^{kt} + \sum_{i=1}^{kt} d_i x^{kt-i},
\]

we find:

\[
p_0 T = \sum_{j=1}^{kt-j} p_j B_j + \sum_{j=0}^{kt-1} r_j B_{n-j}
\]

where

\[
p_j = 1 + \sum_{i=1}^{kt-j} d_i \quad \text{and} \quad r_j = \sum_{i=j+1}^{kt} d_i.
\]

Since \( \phi(E)G_j = 0 \) and \( B_j = j^{t} G_j \), one can easily obtain \( T \) in terms of \( G_1, \ldots, G_{k-1}; G_{n-k+2}, \ldots, G_n \).

The assumption that unity not be a root of the characteristic polynomial has been critical to our discussion so far. We now assume \( \{ G_j \} \) satisfies

\[
X(E) G_j = 0
\]

where \( X(E) \) is a polynomial with \( X(1) = 0 \). Factoring out all the factors \( x - 1 \) in \( X(x) \), we obtain

\[
X(x) = (x - 1)^n \phi(x), \text{ where } \phi(1) \neq 0.
\]
Letting \( C_j = \phi(E)G_j \), we note:

\[
\phi(1)S = \sum_{j=1}^{n} G_j = \sum_{j=1}^{k} G_j q_j + \sum_{j=1}^{n-k} C_j + \sum_{j=0}^{k-1} G_{n-j} s_j,
\]

where

\[
\chi(x) = x^k + \sum_{i=1}^{k} c_i x^{k-i},
\]

\[
q_j = 1 + \sum_{i=1}^{k-j} c_i \text{ and } s_j = \sum_{i=j+1}^{k} c_i.
\]

However, it is known ([2], pp. 548-552) that if \((E - 1)^a C_j = 0\), then \( C_j \) is a polynomial of degree \( \leq a - 1 \). If we assume the formulas for

\[
\sum_{j=1}^{n} j^p
\]

are known, for \( j \) a positive integer, the only problem remaining is that of determining the polynomial \( C_j = d_0 + d_1 j + \ldots + d_{a-1}^a - 1 \). It is easy to show that the difference operator \( E - 1 \) when applied to a polynomial of degree \( r \) yields a polynomial of degree \( r - 1 \). Therefore \((E - 1)^j C_1 \) involves only \( d_{a-1}, d_{a-2}, \ldots, d_j \) and the system of linear equations on the \( d_i \) obtained by computing \((E - 1)^j C_1, j = 0, 1, 2, \ldots, a - 1\) can clearly be solved for the \( d_i \).

The above is a generalization of the technique used by Erbacher and Fuchs to solve problem H-17. [4]

**Example:** Assume that for each positive integer \( j \), \( G_j \) satisfies \( \chi(E)G_j = 0 \), where \( \chi(x) = (x - 1)^3 (x^3 - 3x^2 + 4x + 2) = (x - 1)^3 \phi(x) \). If \( G_1 = G_2 = G_3 = G_4 = G_5 = 0 \), \( G_6 = 1 \), then \( C_1 = \phi(E)G_1 = 0 \), \( C_2 = \phi(E)G_2 = 0 \), \( C_3 = \phi(E)G_3 = 1 \). With \( C_j = d_0 + d_1 j + d_2 j^2 \), we find \((E - 1)^2 C_1 = 2d_2 = 1\), \((E - 1) C_1 = d_1 + 3d_2 = 0\) and \( C_1 = d_0 + d_1 + d_2 \). Hence \( C_j = 1 - (3/2) j + j^2/2 \) and
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\[ \phi(1) S = 4S = 2G_1 - 2G_2 + G_3 + \sum_{j=1}^{n-3} (1 - (3j)/2 + j^2/2) + 3G_n + 6G_{n-1} + 2G_{n-2} \]

\[ = \sum_{j=1}^{n-3} (1 - (3j)/2 + j^2/2) + 3G_n + 6G_{n-1} + 2G_{n-2}. \]

In conclusion, we have seen how the elementary method used to sum a geometric progression can be generalized to find the sum of the first \( n \) terms of a sequence which satisfies a linear homogeneous recursion relation. It may be worth stating that this method is applicable to a sequence whose terms are products of corresponding terms of sequences each of which satisfy a linear homogeneous recursion relation (see [1] pp. 42-45 for a special case).

We propose as a problem for the reader: Find in closed form the sum of the first \( n \) terms of the sequence \( \{w_n\} : \)

\[ 1, 2, 10, 36, 145, \ldots \]

where \( w_n = F_n G_n \) with \( F_{n+2} = F_{n+1} + F_n \) \( (F_1 = F_2 = 1) \) and \( G_{n+2} = 2G_{n+1} + G_n \) \( (G_1 = 1, G_2 = 2) \).

REFERENCES