CONTINUED FRACTION CONVERGENTS AS A SOURCE OF FIBONACCI AND LUCAS IDENTITIES

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Properties of the convergents of continued fractions can be used to develop a number of Fibonacci and Lucas identities. Since references for continued fractions are so commonly available, only those properties of continued fractions necessary to the development of this paper are presented.

Let $\{a_i, b_i\}$ be a sequence of real numbers where $a_0 = 1$, b_0 may be zero, and all the other a_i and b_i are not zero. Then, the continued fraction is given by

$$(1) \qquad X = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \dots}}}}$$

The convergent to X after i terms is given by

$$(2) \qquad \frac{A_i}{B_i} = \frac{b_i A_{i-1} + a_i A_{i-2}}{b_i B_{i-1} + a_i B_{i-2}}$$

for $i = 2, 3, 4, \dots$, $A_0 = b_0$, $B_0 = 1$, $B_1 = b_1$, and $A_1 = b_0 b_1 + a_1$. (In the special case that $a_i = b_i = 1$ for all i, $A_n = F_{n+2}$ and $B_n = F_{n+1}$, where F_n is the nth Fibonacci number.)

It is known that the difference between two successive convergents is

$$(3) \qquad \frac{A_i}{B_i} - \frac{A_{i-1}}{B_{i-1}} = \frac{(-1)^{i-1} a_1 a_2 \dots a_i}{B_i B_{i-1}} .$$

Next, let $S = u_0 + u_1 + u_2 + \ldots$ be any series with partial sums $S_0 = u_0$, $S_1 = u_0 + u_1$, $\ldots$, $S_i = u_0 + u_1 + u_2 + \ldots + u_i$, and set $S_i = A_i/B_i$ for all i. Since $S_i - S_{i-1} = u_i$, from Equation (3), $u_i = (-1)^{i-1} a_1 a_2 \ldots a_i / B_i B_{i-1}$, yielding $a_i = -B_i u_i / B_{i-2} u_{i-1}$ and $b_i = (u_{i-1} + u_i)B_i / u_{i-1} B_{i-1}$. Substituting these values for a_i and b_i into (1) will give the continued fraction representation of S_i below, but the result is very cumbersome to evaluate. The partial sum S_i can be written in the simple form

$$(4)\qquad S_i = u_0 + \cfrac{u_1}{1 - \cfrac{u_2}{(u_1 + u_2) - \cfrac{u_1 u_3}{(u_2 + u_3) - \cdots \cfrac{}{\cdots - \cfrac{u_{i-2}u_i}{(u_{i-1} + u_i)}}}}} \;.$$

The development thus far is found in various standard sources dealing with continued fractions. At last, we have reached the point of departure for the promised Fibonacci and Lucas number representations.

Set $u_i = F_i$, the i-th Fibonacci number defined by $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$. Then, since $F_{i+2} = 1 + (F_1 + F_2 + F_3 + \ldots + F_i) = 1 + S_i$,

$$(5)\qquad F_{i+2} = F_2 + \cfrac{F_1}{F_2 - \cfrac{F_2}{F_3 - \cfrac{F_1 F_3}{F_4 - \cdots \cfrac{}{\cdots - \cfrac{F_{i-2}F_i}{(F_{i-1} + F_i)}}}}}$$

For example,

$$F_6 = F_2 + \cfrac{F_1}{F_2 - \cfrac{F_2}{F_3 - \cfrac{F_1F_3}{F_4 - \cfrac{F_2F_4}{F_5}}}} = 1 + \cfrac{1}{1 - \cfrac{1}{2 - \cfrac{1\cdot 2}{3 - \cfrac{1\cdot 3}{5}}}} = 8.$$

Similarily, if we set $u_i = L_i$, the i-th Lucas number defined by $L_1 = 1$, $L_2 = 3$, $L_{n+1} = L_{n-1} + L_n$, we can write an analogous expression by replacing each F with an L in the above continued fraction representation.

Equation (2) provides

$$(6) \qquad b_i = (A_iB_{i-2} - B_iA_{i-2})/(A_{i-1}B_{i-2} - B_{i-1}A_{i-2})$$

$$= \left(\frac{A_i}{B_i} - \frac{A_{i-2}}{B_{i-2}}\right)\left(\frac{B_i}{B_{i-1}}\right) \Big/ \left(\frac{A_{i-1}}{B_{i-1}} - \frac{A_{i-2}}{B_{i-2}}\right) .$$

As above, let $u_i = F_i$ so that $S_i = A_i/B_i = F_{i+2} - F_2$, and comparing Equations (1) and (5) observe that $b_i = F_{i+1}$. Then, from (6),

$$F_{i+1} = \left[(F_{i+2} - F_2) - (F_i - F_2)\right]\cdot B_i/B_{i-1}\cdot\left[(F_{i+1} - F_2) - (F_i - F_2)\right]$$

which reduces at once to $B_i = B_{i-1}F_{i-1}$. Then, the equation above can be written as

$$F_{i+1} = (F_{i+2} - F_i)F_{i-1}/(F_{i+1} - F_i)$$

which becomes

$$F_{i+2}F_{i-1} = F_{i+1}^2 - F_i^2$$

or

$$F_{i+1}^2 - F_{i+1}F_i - (F_{i+2} - F_i)F_{i-1} = 0.$$

The second form has solution

(7) $$2F_{i+1} = F_i \pm \sqrt{F_i^2 + 4\,F_{i-1}\,(F_{i+2} - F_i)}$$

where obviously the radicand must be the square of a positive integer. Taking trial values $i = 5$ and $i = 6$ leads to $11^2 = L_5^2$ and $18^2 = L_6^2$, and suggests

(8) $$F_i^2 + 4\,F_{i-1}\,F_{i+1} = L_i^2,$$

which can be established by mathematical induction. Taking the positive sign in (7) gives

$$2\,F_{i+1} = F_i + L_i \quad \text{or} \quad L_i = F_{i+1} + F_{i-1},$$

a well-known result.

A parallel development can be used for the Lucas numbers leading to

$$L_{i+2}\,L_{i-1} = L_{i+1}^2 - L_i^2,$$

$$L_{i+1}^2 - L_{i+1}L_i - (L_{i+2} - L_i)L_{i-1} = 0$$

with solution

(9) $$2\,L_{i+1} = L_i \pm \sqrt{L_i^2 + 4\,L_{i-1}\,(L_{i+2} - L_i)}\ .$$

By using the identity $L_i = F_{i+1} + F_{i-1}$, the radicand can be reduced to $25F_i^2$, leading to the parallel of Equation (8),

(10) $$L_i^2 + 4\,L_{i-1}\,L_{i+1} = 25F_i^2.$$

As a side benefit, combining Equations (8) and (10) gives us

$$6\,F_i^2 = L_{i-1}L_{i+1} + F_{i-1}F_{i+1},$$

and substituting $25F_i^2$ for the radicand in Equation (9) yields

$$5F_i = L_{i-1} + L_{i+1}.$$

Returning to Equation (3) and solving for a_i, we have

$$-a_i = (A_i B_{i-1} - B_i A_{i-1})/(A_{i-1}B_{i-2} - B_{i-1}A_{i-2})$$

$$= \left(\frac{A_i}{B_i} - \frac{A_{i-1}}{B_{i-1}}\right)\left(\frac{B_i}{B_{i-2}}\right) \Big/ \left(\frac{A_{i-1}}{B_{i-1}} - \frac{A_{i-2}}{B_{i-2}}\right) .$$

Comparing Equations (1) and (5) shows $-a_i = F_i F_{i-2}$, so that

$$(11)\quad F_i F_{i-2} = \left[(F_{i+2} - F_2) - (F_{i+1} - F_2)\right] B_i / B_{i-2} \left[(F_{i+1} - F_2) - (F_i - F_2)\right]$$

$$= (F_{i+2} - F_{i+1})(F_{i-1}F_{i-2})/(F_{i+1} - F_i).$$

Simplifying, we have

$$F_i^2 - F_i F_{i+1} + (F_{i+2} - F_{i+1})F_{i-1} = 0$$

with solution

$$2F_i = F_{i+1} \pm \sqrt{F_{i+1}^2 - 4F_{i-1}(F_{i+2} - F_{i+1})}$$

$$= F_{i+1} \pm \sqrt{F_{i+1}^2 - 4F_{i-1}(F_i)} \quad .$$

Replacing F_{i+1}^2 by $(F_i + F_{i-1})^2$ leads to

$$F_{i+1}^2 - 4F_i F_{i-1} = F_{i-2}^2$$

so that the equation above becomes

$$2F_i = F_{i+1} + F_{i-2}.$$

The Lucas number equivalents are found by replacing each F by an L from Equation (11) onwards.

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