ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications concerning Elementary Problems and Soltuions to Prof. A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within two months of publication.

B-76 *(Originally P-1 of this Quarterly, Vol. 1, No. 2, p. 74)*
Proposed by James A. Jeske, San Jose State College, San Jose, California.

The recurrence relation for the sequence of Lucas numbers is $L_{n+2} - L_{n+1} - L_n = 0$ with $L_1 = 1$, $L_2 = 3$.
Find the transformed equation, the exponential generating function, and the general solution.

B-77 *(Originally P-2 of this Quarterly, Vol. 1, No. 2, p. 74)*
Proposed by James A. Jeske, San Jose State College, San Jose, California.

Find the general solution and the exponential generating function for the recurrence relation

$$y_{n+3} - 5y_{n+2} + 8y_{n+1} - 4y_n = 0,$$

with $y_0 = 0$, $y_1 = 0$, and $y_2 = -1$.

B-78 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Show that

$$F_n = L_{n-2} + L_{n-6} + \cdots + L_{n-2-4m} + e_n, \quad n > 2,$$

where m is the greatest integer in $(n - 3)/4$, and $e_n = 0$ if $n \equiv 0 \pmod 4$, $e_n = 1$ if $n \not\equiv 0 \pmod 4$.

B-79 *Proposed by Brother U. Alfred, St. Mary's College, St. Mary's College, California*

Let $a = (1 + \sqrt{5})/2$. Determine a closed expression for

$$X_n = [a] + [a^2] + \ldots + [a^n]$$

where the square brackets mean "greatest integer in."

B-80 *Proposed by Maxey Brooke, Sweeny, Texas*

Solve the division alphametic

$$\begin{array}{r} \text{PISA} \\ \text{FIB}\,\overline{)\,\text{XONACCI}} \end{array}$$

where each letter represents one of the nine digits 1, 2, ..., 9 and two letters may represent the same digit.

B-81 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Prove that only one of the Fibonacci numbers 1, 2, 3, 5, ... is a prime in the ring of Gaussian integers.

SOLUTIONS

A LUCAS NUMBERS IDENTITY

B-64 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that $L_n L_{n+1} = L_{2n+1} + (-1)^n$, where L_n is the n-th Lucas number defined by $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$.

Solution by John Allen Fuchs, University of Santa Clara, Santa Clara, California

By the Binet formula

$$L_n = a^n + b^n$$

where $a = (1+\sqrt{5})/2$ and $b = (1-\sqrt{5})/2$ and $ab = -1$. Then

$$L_n L_{n+1} = (a^n + b^n)(a^{n+1} + b^{n+1}) = a^{2n+1} + a^n b^{n+1} + a^{n+1} b^n + b^{2n+1}$$

$$= a^{2n+1} + b^{2n+1} + (ab)^n (a+b) = L_{2n+1} + (-1)^n \ .$$

Also solved by John E. Homer, Jr.; Douglas Lind; Benjamin Sharpe; M. N. Srikanta Swamy; John Wessner; and the Proposer

OPERATORS

B-65 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Let u_n and v_n be sequences satisfying $u_{n+2}+au_{n+1}+bu_n = 0$ and $v_{n+2}+cv_{n+1}+dv_n = 0$ where a, b, c, and d are constants and let $(E^2+aE+b)(E^2+cE+d) = E^4+pE^3+qE^2+rE+s$. Show that $y_n = u_n + v_n$ satisfies

$$y_{n+4}+py_{n+3}+qy_{n+2}+ry_{n+1}+sy_n = 0 \ .$$

Solution by David Zeitlin, Minneapolis, Minnesota

Let $P(E) = E^2 + aE + b$ and $Q(E) = E^2 + cE + d$, where $P(E)u_n = 0$, $Q(E)v_n = 0$, $P(E)0 = 0$, and $Q(E)0 = 0$. Since $P(E)Q(E) \equiv Q(E)P(E)$ we have

$$P(E)Q(E)(u_n + v_n) = Q(E)\,[P(E)u_n] + P(E)0 = Q(E)0 = 0,$$

which is the desired result.

Also solved by Douglas Lind; M.N.S. Swamy; and the proposer

B-66 *Proposed by D. G. Mead, University of Santa Clara, Santa Clara, California*

Find constants p, q, r, and s such that

$$y_{n+4}+py_{n+3}+qy_{n+2}+ry_{n+1}+sy_n = 0$$

is a 4th order recursion relation for the term-by-term products $y_n = u_n v_n$ of solutions of $u_{n+2}-u_{n+1}-u_n = 0$ and $v_{n+2}-2v_{n+1}-v_n = 0$.

Solution by Jeremy C. Pond, Sussex, England

$u_n = Aa^n + Bb^n$ where a, b are the roots of $x^2 - x - 1 = 0$ and $v_n = Cc^n + Dd^n$ where c, d are the roots of $x^2 - 2x - 1 = 0$. Thus $y_n = AC(ac)^n + AD(ad)^n + BC(bc)^n + BD(bd)^n$, and so ac, ad, bc, bd are the solutions of

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

i.e.,

$$p = -(a + b)(c + d) = -2$$

$$q = b^2cd + abd^2 + 2abcd + abc^2 + a^2cd$$

$$= (a + b)^2cd + (c + d)^2ab - 2abcd = -1 -4 -2 = -7$$

$$r = -abcd(bd + bc + ad + ac) = -abcd(a + b)(c + d) = -2$$

$$s = (abcd)^2 = 1.$$

Summarizing: $p = -2$; $q = -7$; $r = -2$; $s = 1$.

Also solved by Douglas Lind; M.N.S. Swamy, David Zeitlin; and the proposer

B-67 *Proposed by D. G. Mead, University of Santa Clara, Santa Clara, California*

Find the sum $1 \cdot 1 + 1 \cdot 2 + 2 \cdot 5 + 3 \cdot 12 + \ldots + F_nG_n$, where $F_{n+2} = F_{n+1} + F_n$ and $G_{n+2} = 2G_{n+1} + G_n$.

Solution by M.N.S. Swamy, University of Saskatchewan, Regina, Canada

Using the result of Problem B-66, we have the recurrence relation,

$$y_{n+4} - 2y_{n+3} - 7y_{n+2} - 2y_{n+1} + y_n = 0 \qquad (1)$$

where, $y_n = F_nG_n$.

Substituting successively $1, 2, \ldots, n$ for n in (1) and adding we get

$$(y_n + y_2 + \ldots + y_n) - 2y_2 - 9y_3 - 11y_4 - 10(y_5 + \ldots + y_{n+1})$$

$$-8y_{n+2} - y_{n+3} + y_{n+4} = 0$$

or

$$9 \sum_1^n y_r = (10y_1 + 8y_2 + y_3 - y_4) - 10y_{n+1} - 8y_{n+2} - y_{n+3} + y_{n+4} \quad .$$

Now, $10y_1 + 8y_2 + y_3 - y_4 = 10 + 8 \cdot 1 \cdot 2 + 2 \cdot 5 - 3 \cdot 12 = 0.$
Hence,

$$9 \sum_1^n y_r = -10y_{n+1} - 8y_{n+2} - y_{n+3} + y_{n+4} \quad .$$

Substituting for y_{n+4} from (1), the above equation reduces to

$$9 \sum_1^n y_r = y_{n+3} - y_{n+2} - 8y_{n+1} - yn.$$

Again using (1), this may to reduced to

$$9 \sum_1^n y_r = y_{n+2} - y_{n+1} + y_n - y_{n-1}.$$

Therefore we have

$$1 \cdot 1 + 1 \cdot 2 + 2 \cdot 5 + 3 \cdot 12 + \ldots + F_n \cdot G_n$$

$$= (F_{n+2}G_{n+2} - F_{n+1}G_{n+1} + F_nG_n - F_{n-1}G_{n-1})/9.$$

Also solved by Douglas Lind, Jeremy C. Pond, David Zeitlin, and the proposer. Pond and Zeitlin simplified the sum to the form $(F_{n+1}G_n + F_nG_{n+1})/3$.

FIBONACCI DIMENSIONS FOR PARALLELEPIPEDS

B-68 *Proposed by Walter W. Horner, Pittsburgh, Pennsylvania*

Find expressions in terms of Fibonacci numbers which will generate integers for the dimensions and diagonal of a rectangular parallelepiped, i.e., solutions of

$$a^2+b^2+c^2 = d^2 \quad .$$

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

Let F_r and F_s be any two Fibonacci numbers of opposite parity. Then

$$F_r^2 + F_s^2 = 2k + 1 = (k+1)^2 - k^2.$$

Since $k = \frac{1}{2}(F_r^2 + F_s^2 - 1)$, an expression of the desired type is

$$F_r^2 + F_s^2 + \left(\frac{F_r^2 + F_s^2 - 1}{2}\right)^2 = \left(\frac{F_r^2 + F_s^2 + 1}{2}\right)^2 .$$

Also solved by the proposer

SIMULTANEOUS EQUATIONS

B-69 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Solve the system of simultaneous equations:

$$xF_{n+1} + yF_n = x^2 + y^2$$

$$xF_{n+2} + yF_{n+1} = x^2 + 2xy$$

where F_n is the n-th Fibonacci number.

Solution by Jeremy C. Pond, Sussex, England

It is easy to check two solutions:

(a) $x = 0$ and $y = 0$

(b) $x = F_{n+1}$ and $y = F_n$.

Now from the second equation: $y = x(x - F_{n+2})/(F_{n+1} - 2x)$ unless $F_{n+1} = 2x$. This special case leads us to (a) and (b) with $n = -1$.

Substitute this expression for y in the first equation and multiply by $(F_{n+1} - 2x)^2$. This leads to

$$x(F_{n+1} - x)(F_{n+1} - 2x)^2 = x(x - F_{n+2})(x^2 - xF_{n+2} - F_nF_{n+1} + 2xF_n) .$$

One solution is $x = 0$ and the others satisfy:

$$(x - F_{n+1})(F_{n+1} - 2x)^2 + (x - F_{n+2})(x^2 - xF_{n-1} - F_nF_{n+1}) = 0 .$$

This is a cubic with three solutions. It is easy to verify that the sum of these two roots is $2F_{n+1}$ and the product is $(-1)^nF_{n+1}/5$.

We know that one of these solutions is F_{n+1} so the other two have sum F_{n+1} and products $(-1)^n/5$; i.e. they are:

$$(F_{n+1} \pm \sqrt{F_{n+1}^2 + [4(-1)^{n+1}/5]})/2 = \frac{\alpha^{n+1}}{\sqrt{5}}, -\frac{\beta^{n+1}}{\sqrt{5}}$$

Thus the complete solution of the system of equations is

(a) $x = 0;\ y = 0$

(b) $x = F_{n+1};\ y = F_n$

(c) and (d)

$$x = (F_{n+1} \pm \sqrt{F_{n+1}^2 + [4(-1)^{n+1}/5]})/2 = \frac{\alpha^{n+1}}{\sqrt{5}}, -\frac{\beta^{n+1}}{\sqrt{5}}$$

$$v = \frac{\alpha^n}{\sqrt{5}}, -\frac{\beta^n}{\sqrt{5}}$$

Also solved by M. N. S. Swamy and the proposer

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