GAUSSIAN FIBONACCI AND LUCAS NUMBERS

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Recently A. F. Horadam [2] introduced the concept of the complex Fibonacci numbers and established some quite general identities concerning them. It is the purpose of this paper to consider merely two of the Complex Fibonacci sequences and extend some relationships which are known about the common Fibonacci sequences to the Complex Fibonaccies.

Def. 1: The Gaussian Fibonacci sequence is $GF_0 = i$; $GF_1 = 1$; $GF_n = GF_{n-1} + GF_{n-2}$ for $n > 1$. It is easy to see that $GF_n = F_n + F_{n-1}\, i$.

Def. 2: The Gaussian Lucas sequence is $GL_0 = 2-i$; $GL_1 = 1+2i$; $GL_2 = 3+i$ $GL_n = GL_{n-1} + GL_{n-2}$ for $n > 2$. It is easy to see that $GL_n = L_n + L_{n-1} i$.

Analogous to the usual identities stated by S. L. Basin and V. E. Hoggatt, Jr. [1], the following identities are easily attainable.

For $n \geq 2$

(1) $$\sum_{j=o}^{n} GF_j = GF_{n+2^{-1}}$$

(2) $$\sum_{j=o}^{n} GL_j = GL_{n+2} - (1 + 2i)$$

(3) $$GF_{n+1}\, GF_{n-1} - GF_n^{\ 2} = (-1)^n (2-i)$$

(4) $$GL_{n+1} GL_{n-1} - GL_n^{\ 2} = (-1)^{n+1}\, 5(2-i)$$

(5) $$GL_n = GF_{n+1} + GF_{n-1}$$

(6) $$GF_{n+1}^{\,2} + GF_n^{\,2} = F_{2n}(1 + 2i)$$

(7) $$GF_{n+1}^{\,2} - GF_{n-1}^{\,2} = F_{2n-1}\,(1 + 2i)$$

(8) $$GF_n GL_n = F_{2n-1}\,(1 + 2i)$$

(9) $$GF_{n+1} GF_{p+1} + GF_n\, GF_p = F_{n+p}\,(1 + 2i)$$

(10) $$\sum_{j=1}^{n} GF_j^{\,2} = F_n^{\,2}\,(1 + 2i) + (-1)^n\, i - i$$

(11) $$GL_n^{\,2} - 5\, GF_n^{\,2} = (-1)^n\, 4(2-i)$$

(12) $$GF_{-n} = iGF_n = i(F_n - F_{n-1}i)$$

Corollary to (11): GL_n is composite for $n \geq 2$.

The occurence of $1 + 2i$, $2 + i$, $(1-2i)$, and $(2-i)$ seems poetic in these formulae in view of the fact they are factors of 5. Some of the usual results mentioned in Vorob'ev [5] can be extended yielding

$$\sum_{j=1}^{n} GF_{2j-1} = GF_{2n} - i$$

$$\sum_{j=1}^{n} GF_{2j} = GF_{2n+1} - 1$$

$$\sum_{j=1}^{2n} (-1)^j\, GF_j = GF_{2n-1} - 1 + i$$

$$\sum_{j=1}^{n} (-1)^j\, GF_j = (-1)^{j+1}\quad GF_n - 1 + i$$

The norm of the Gaussian Fibonacci is $N(GF_n) = F_n^2 + F_{n-1}^2 = F_{2n-1}$.

A well known theorem mentioned in Hardy and Wright [3] is

Theorem A: For $n \geqq 2$, $F_n \mid F_m$ if and only if $n \mid m$

And a theorem mentioned recently by G. Michael [4] is

Theorem B: $(F_n, F_m) = F_{(n,m)}$.

The corresponding result for Theorem A with Gaussian Fibonacci numbers is

Theorem 1: For $n > 2$, $GF_n \mid FG_m$ if and only if $2n-1 \mid 2m-1$, divisibility in the sense of Gaussian Integers.

We start with the following preliminary.

Lemma: If $2n-1 \mid 2m-1$ then $2n-1 \mid m + n-1$.

Proof: It follows that if $2n-1 \mid 2m-1$ then $2n-1 \mid 2m-1 - (2n-1) = 2m-2n$. Now $(2, 2n-1) = 1$ since $2n-1$ is odd therefore $2n-1 \mid m-n$. It now follows that $2n-1 \mid (2m-1) - (m-n) = m+n-1$.

Proof of the Theorem 1: A necessary condition for $GF_n \mid GF_m$ is that $N(GF_n) \mid N(GF_m)$. But this happens only when $F_{2n-1} \mid F_{2m-1}$ or by Theorem A only when $2n-1 \mid 2m-1$. Therefore one concludes that a necessary condition for $GF_n \mid GF_m$ is that $2n-1 \mid 2m-1$.

On the other hand if $2n-1 \mid 2m-1$ then $N(GF_n) = F_{2n-1} \mid F_{2m-1} = N(GF_m)$. This means that $N(GF_m/GF_n)$ is a positive integer. Now

$$\frac{GF_m}{GF_n} = \frac{F_m + F_{m-1}\, i}{F_n + F_{n-1}\, i}$$

$$= \frac{F_m F_n + F_{m-1} F_{n-1} + (F_{m-1} F_n - F_{n-1} F_m)\, i}{F_n^2 + F_{n-1}^2}$$

$$= \frac{F_m F_n + F_{m-1} F_{n-1}}{F_{2n-1}} + \frac{F_{m-1} F_n - F_{n-1} F_m\, i}{F_{2n-1}}$$

$$= \frac{F_{m+n-1}}{F_{2n-1}} + \frac{F_{m-1} F_n - F_{n-1} F_m\, i}{F_{2n-1}}$$

But by the lemma and Theorem A it follows that $F_{2n-1} \mid F_{m+n-1}$.

Hence F_{m+n-1} / F_{2n-1} is an integer a. It follows that

$$\frac{F_{m-1} F_n - F_{n-1} F_m}{F_{2n-1}}$$

must also be an integer, b, since the norm is an integer. Therefore $GF_m/GF_n = a+b\,i$. Q. E. D.

The following interesting by-product has been established.

Corollary: For $n \geq 2$, $F_{2n-1} \mid F_{m-1} F_n - F_{n-1} F_m$ if and only if $2n-1 \mid 2m-1$.

Def. 3: If z and w are Gaussian Integers and the greatest common divisor of z and w is that Gaussian Integer y such that $y \mid z$ and $y \mid w$ and if $t \mid z$ and $t \mid w$ then $N(t) \leq N(y)$. Notationwise $(z, w) = y$.

The analogy to Theorem B is as follows:

Theorem 2: $(GF_m, GF_n) = GF_k$ where $2k-1 = (2m-1, 2n-1)$.

Proof: Since 2k-1 divides 2m-1 and also 2n-1 it follows from Theorem 1 that $GF_k \mid GF_m$ and $GF_k \mid GF_n$: If $H \mid GF_m$ and $H \mid GF_n$ then $N(H) \mid N(GF_m) = F_{2m-1}$ and $N(H) \mid N(GF_n) = G_{2n-1}$. Now by Theorem B $(F_{2m-1}, F_{2n-1}) = F_{(2m-1,\ 2n-1)} = F_{2k-1}$. Now $N(H) \mid F_{2k-1} = N(GF_k)$ hence $N(H) \leq N(GF_k)$. Q. E. D.

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