ON A GENERATING FUNCTION ASSOCIATED WITH GENERALIZED FIBONACCI SEQUENCES

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Let p, q, a and b be complex numbers and assume that $q \neq 0$ and $q \neq p^2$. Let the sequence $u_n(p, q; a, b)$ be the solution of the recurrence relation

(1) $$u_n = 2pu_{n-1} - qu_{n-2}, \quad n \geq 2,$$

with the "initial condition"

(2) $$u_0 = a, \quad u_1 = b + pa.$$

Here and below we omit arguments whenever they are obvious.

If $p = 1/2$ and $q = -1$, the above sequence reduces to $u_n(1/2, -1; a, b) = H_n$, the generalized Fibonacci sequence. Further, $u_n(1/2, -1; 0, 1) = F_n$, the Fibonacci sequence, and $u_n(1/2, -1; 2, 0) = L_n$, the Lucas sequence.

For any integer k, define the function $x \longrightarrow u^{(k)}(x; p, q; a, b) = u^{(k)}(x)$ by the formula

(3) $$u^{(k)}(x) = \sum_{0 \leq n \leq \infty} (u_n)^k x^n.$$

Since, as is easily verified, $u_n \leq A\, s^n$ where $s = |p| + \sqrt{|p|^2 + |q|}$, the series in (3) converges at least for $x < s^{-k}$. A few years ago Carlitz [1] summed the series for $u^{(k)}$ in special cases when $a = 1$, $b = p$ (using the present notation) and $a = 2$, $b = 0$. For related results see also the papers by Gould [2] and Riordan [3]. A. F. Horadam recently studies [(3)] the generating functions $u^{(k)}$ and indicated that they can be summed by using methods analogous to those employed by Carlitz, which are rather complicated. The objective of this paper is to give a straightforward derivation of a formula for $u^{(k)}$ with any a and b.

<u>Theorem.</u> <u>If</u> $u_n(p, q; a, b)$ <u>is defined by (1-2) and</u> $u^{(k)}$ <u>is defined by (3), then</u>

$$u^{(k)}(x) = \sum_{0 \le \gamma < k/2} \frac{A_{\gamma,k} + q^{\gamma}[B_{\gamma,k}u_{k-2\gamma}(p,q;0,1) - A_{\gamma,k}u_{k-2\gamma}(p,q;1,0)]x}{1 - 2q^{\gamma}u_{k-2\gamma}(p,q;1,0)x + q^k x^2}$$

$$+ \frac{(1 + (-1)^k)A_{k/2,k}}{4[1 - q^{k/2}x]},$$

where $A_{\gamma,k}$ and $B_{\gamma,k}$ with $\gamma \le [k/2]$ are homogeneous forms in a, and b of degree k defined by

$$A_{\gamma,k} = 2^{1-k}\binom{k}{\gamma}(a^2 - \beta)^{\gamma} \sum_{0 \le 2j \le k-2\gamma} \binom{k-2\gamma}{2j} a^{k-2\gamma-2j}\beta^j \quad .$$

$$B_{\gamma,k} = 2^{1-k}\binom{k}{\gamma}(a^2 - \beta)^{\gamma} b \sum_{0 \le 2j+1 \le k-2\gamma} \binom{k-2\gamma}{2j+1} a^{k-2\gamma-2j-1}\beta^j \quad ,$$

with $\beta = b^2/(p^2 - q)$.

Note that in the last term in the formula for $u^{(k)}(x)$ the first factor is zero if k is odd so that we should not be concerned by the fact that $A_{k/2,k}$ is not defined when k is odd.

Our proof exploits the fact that the zeros of $z^2 - 2z\cos\theta + 1$, with any θ real or complex, are $e^{i\theta}$ and $e^{-i\theta}$ whose powers are easily managed. Let α and θ be such that

(4) $$\alpha^2 = q, \qquad p = \alpha\cos\theta \; .$$

Since $q \ne 0$ and $p \ne q^2$, $\alpha \ne 0$ and $\cos\theta \ne \pm 1$. Since the function $z \longrightarrow \cos z$ is onto the complex plane, a number θ satisfying (4) exists; it may be, or course, a complex number. Note that $\alpha^2\sin^2\theta = \alpha^2 - \alpha^2\cos^2\theta = q - p^2 \ne 0$.

Set $u_n = \alpha^n v_n$. Then $v_n = 2(\cos\theta)v_{n-1} - v_{n-2}$ $(n \ge 2)$ from which it follows, by using well known results for linear recurrences with constant coefficients, that $v_n = s\,e^{in\theta} + t\,e^{-in\theta}$ with some s and t which are determined by the "initial conditions" (2). We now conclude that

(5) $$u_n(p, q; a, b) = \alpha^n(s\, e^{in\theta} + t\, e^{-in\theta}) \ .$$

Setting $n = 0, 1$ in succession we get

(6') $$s + t = a$$

and $\alpha(\cos\theta)(s + t) + i\alpha(\sin\theta)(s - t) = b + pa$, whence it follows, on using (6') and (4), that

(6'') $$s - t = b/(i\alpha \sin\theta).$$

The expressions for s and t may be easily obtained but will not be needed here. On the other hand we note that if $s = t = 1/2$ then $a = 1$ and $b = 0$, while if $s = -t = 1/2$ then $a = 0$ and $b = i\alpha \sin\theta$. Thus it follows from (5) that

(7) $$\begin{cases} \alpha^n \cos n\theta = u_n(p, q; 1, 0), \\ \alpha^n \sin n\theta = \alpha(\sin\theta) u_n(p, q; 0, 1), \end{cases}$$

identifications which will be used in the sequel. [4]

We are now ready for the evaluation of $u^{(k)}(x)$. Using the binomial theorem, we get

(8) $$(s\, e^{in\theta} + t\, e^{-in\theta})^k = \sum_{0 \le \gamma \le k} \binom{k}{\gamma} s^{\gamma} t^{k-\gamma} e^{in(2\gamma - k)\theta}$$

$$= \sum_{0 \le \gamma \le k/2} \binom{k}{\gamma} (st)^{\gamma} (s^{k-2\gamma} e^{in(k-2\gamma)\theta} + t^{k-2\gamma} e^{-in(k-2\gamma)\theta})$$

$$+ 2^{-1}(1 + (-1)^k) \binom{k}{k/2} (st)^{k/2} ,$$

where the last equality follows by pairing off terms with γ and $k - \gamma$, and where the last term is to be set equal to zero if k is odd. On substituting (5) in (3), using (8), interchanging the order of summation, and finally summing geometric series we obtain

(9)
$$u^{(k)}(x) = \sum_{0 \le \gamma < k/2} \binom{k}{\gamma}(st)^\gamma \sum_{n=0}^{\infty} [s^{k-2\gamma}(xa^k e^{i(k-2\gamma)\theta})^n$$

$$+ t^{k-2\gamma}(xa^k e^{-(k-2\gamma)\theta})^n]$$

$$+ 2^{-1}(1 + (-1)^k)\binom{k}{k/2}(st)^{k/2} \sum_{n=0}^{\infty} (xa^k)^n$$

$$= \sum_{0 \le \gamma < k/2} \binom{k}{\gamma}(st)^\gamma \left[\frac{s^{k-2\gamma}}{1 - xa^k e^{i(k-2\gamma)\theta}} + \frac{t^{k-2\gamma}}{1 - xa^k e^{-i(k-2\gamma)\theta}}\right]$$

$$+ 2^{-1}(1 + (-1)^k)\binom{k}{k/2} \frac{(st)^{k/2}}{1 - xa^k} \quad .$$

Observing that $a^{2k} = q^k$, $a^k = q^{k/2}$ if k is even, $a^k\cos(k-2\gamma)\theta = q^\gamma u_{k-2\gamma}(p, q; 1, 0)$ and $a^k\sin(k-2\gamma) = q^\gamma a(\sin\ \theta) u_{k-2\gamma}(p, q; 0, 1)$ if $2\gamma < k$, see formulae (7), the form for $u^{(k)}(x)$ asserted in the theorem follows from (9) if we define

(10)
$$A_{\gamma,k} = \binom{k}{\gamma}(st)^\gamma [s^{k-2\gamma} + t^{k-2\gamma}], \qquad 2\gamma \le k,$$

$$B_{\gamma,k} = i\binom{k}{\gamma}(st)^\gamma [s^{k-2\gamma} - t^{k-2\gamma}]a \sin \theta, \qquad 2\gamma < k.$$

It remains to evaluate $A_{\gamma,k}$ and $B_{\gamma,k}$ in terms of a and b. Let $\beta = [b/(ia \cos\theta)]^2 = b^2/(p^2 - q)$. From (6') and (6'') we get:

$$st = (a^2 - \beta)/4,$$

whence $(st)^\gamma = 2^{-2\gamma}(a^2 - \beta)^\gamma$,

$$s^m + t^m = 2^{-m}([(s+t)+(s-t)]^m + [(s+t)-(s-t)]^m) = 2^{1-m} \sum_{0 \le 2j \le m} \binom{m}{2j} a^{m-2j} \beta^j,$$

$$s^m - t^m = 2^{-m}([(s+t)+(s-t)]^m - [(s+t)-(s-t)]^m)$$

$$= 2^{1-m} \frac{b}{i\alpha \sin\theta} \sum_{0 \le 2j+1 \le m} \binom{m}{2j+1} a^{m-2j-1} \beta^j .$$

Substituting these in (10) we get the stated result. This completes the proof of the theorem.

It might not be superfluous to point out some special cases which may be obtained from the theorem. If $p = 1/2$ and $q = -1$, then $u^{(k)}(x;1/2,-1;a,b) = H^{(k)}(x;a,b)$, the generating function for k^{th} powers of the generalized Fibonacci sequence $H_n(a,b)$. In this case the formulae for $A_{\gamma,k}$ and $B_{\gamma,k}$ do not simplify appreciably except that we have now $\beta = 4b^2/5$, while $u_n(1/2,-1;0,1) = F_n$ and $u_n(1/2,-1;1,0) = L_n/2$. Furthermore, if also $a = 0$ and $b = 1$, then $A_{\gamma,k} = 0$ if k is odd and $B_{\gamma,k} = 0$ if k is even, while $B_{\gamma,k} = (-1)\binom{k}{\gamma} 5^{(1-k)/2}$ if k is odd and $A_{\gamma,k} = 2(-1)\binom{k}{\gamma} 5^{-k/2}$ if k is even. The theorem then yields the well known formulae

$$(11)\begin{cases} F^{(k)}(x) = 5^{(1-k)/2} x \displaystyle\sum_{0 \le \gamma \le (k-1)/2} \frac{\binom{k}{\gamma} F_{k-2\gamma}}{1-(-1)^\gamma L_{k-2\gamma}x - x^2}, & \text{if } k \text{ is odd,} \\ F^{(k)}(x) = 5^{-k/2}\Big[\displaystyle\sum_{0 \le \gamma < k/2} \frac{\binom{k}{\gamma}(2(-1)^\gamma - L_{k-2\gamma}x)}{1-(-1)^\gamma L_{k-2\gamma}x + x^2} + \frac{\binom{k}{k/2}}{(-1)^{k/2} - x}\Big], & \text{if } k \text{ is even,} \end{cases}$$

Lastly, if $p = 1/2$, $q = -1$, $a = 2$ and $b = 0$, we get $A_{\gamma,k} = 2\binom{k}{\gamma}$ and $B_{\gamma,k} = 0$ whence

$$L^{(k)}(x) = \sum_{0 \le \gamma < k/2} \frac{\binom{k}{\gamma}\left(2 - (-1)^{\gamma} L_{k-2\gamma} x\right)}{1-(-1)^{\gamma} L_{k-2\gamma} x + (-1)^k x^2} + \frac{\binom{k}{k/2}}{1-(-1)^{k/2} x} \cdot \frac{1+(-1)^k}{2} . \tag{12}$$

In conclusion we note that by squaring the two equalities in (7) and adding we get the identity

$$q^n = [u_n(p, q; 1, 0)]^2 + (q - p^2)\,[u_n(p, q; 0, 1)]^2 \tag{13}$$

If $p = 1/2$ and $q = -1$, the identity (13) simplifies to the well-known identity

$$4(-1)^n = L_n^2 - 5F_n^2 . \tag{14}$$

REFERENCES

1. Carlitz, L., "Generating Functions for Powers of Certain Sequences of Numbers", Duke Math. Journal, 29(1962) pp. 521-538.
2. Gould, H. W., "Generating Functions for Products of Powers of Fibonacci Numbers", Fibonacci Quarterly, April 1963, pp. 1-16.
3. Riordan, J., "Generating Functions for Powers of Fibonacci Numbers", Duke Math. Journal, 29(1962) pp. 5-12.

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FOOTNOTES

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(3) Oral communication.

(4) Formulae (7) show the connection between u_n and the Chebyshev polynomials. For example, $u_n(p, q; 1, 0) = \alpha^n T_n(p/\alpha)$, where $\alpha^2 = q$.

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