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1. Introduction

In this paper, the notion of Fibonacci word is introduced and the structure of these words is investigated.

Let \mathscr{A} be a nonempty set and let x and y be two words in the alphabet \mathscr{A} . A Fibonacci sequence of words derived from x and y is a sequence of words w_1 , w_2 , w_3 , ... with the property that

$$w_1 = x$$
, $w_2 = y$, $w_{n+1} = w_n w_{n-1}$ or $w_{n-1} w_n$.

The pair x and y are called the *initial words* of the Fibonacci sequence of words; the words $w_n w_{n-1}$ and $w_{n-1} w_n$ are called the *immediate successors* of w_n and w_n is their *immediate predecessor*. We remark that the Fibonacci sequence of words considered by Knuth [3, p. 85] is the one obtained as above by letting

$$w_{n+1} = w_n w_{n-1}$$
 for all $n \ge 3$

and the one considered by Higgins [1] and Turner [4, 5] is obtained by letting

 $w_{n+1} = w_{n-1}w_n$ for all $n \ge 3$.

Let \mathscr{S} be the set of all such sequences of words derived from x and y; let \mathscr{S}_n be the collection of words which happen to be the nth term of some members of $\widetilde{\mathscr{S}}$. For example,

$$\mathscr{S}_{1} = \{x\}, \ \mathscr{S}_{2} = \{y\}, \ \mathscr{S}_{3} = \{yx, xy\}, \ \mathscr{S}_{4} = \{yxy, yyx, xyy\}.$$

Denote the union of \mathcal{G}_n (n = 1, 2, ...) by \mathcal{G} . Members of \mathcal{G}_n (resp. \mathcal{G}) are called the n^{th} Fibonacci words (resp. Fibonacci words). Note that each word has an obvious representation in terms of x and y. Throughout this paper, we consider only such a representation.

Lemma 1: Let w be a Fibonacci word. Then the following statements are true.

- (i) If w starts (resp. ends) with an x, then w cannot end (resp. start) with an x.
- (ii) If w starts (resp. ends) with a y, then w cannot end (resp. start) with a yy.
- (iii) There cannot be three or more consecutive occurrences of y and there cannot be two or more consecutive occurrences of x in w.

Proof: The result is proved by mathematical induction.

Let $\mathscr{S}_n(x, \cdot)$ [resp. $\mathscr{S}_n(\cdot, y)$] denote those n^{th} Fibonacci words which start with an x (resp. end with a y) and let

$$\mathscr{S}_n(x, y) = \mathscr{S}_n(x, \cdot) \cap \mathscr{S}_n(\cdot, y).$$

Define $\mathscr{G}(x, \cdot)$, $\mathscr{G}(\cdot, y)$, etc., in a similar way.

Corollary 2:

$$\begin{aligned} \mathcal{S}_n &= \mathcal{S}_n(x, y) \cup \mathcal{S}_n(y, y) \cup \mathcal{S}_n(y, x) \quad \text{for all } n; \\ \mathcal{S} &= \mathcal{S}(x, y) \cup \mathcal{S}(y, y) \cup \mathcal{S}(y, x). \end{aligned}$$

Using finite binary sequences, let us label the Fibonacci words as follows:

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(1.1)
$$w_1 = x - w_2 = y - w_3^0 = yx - w_4^{00} = yxy - w_4^{01} = yyx - w_4^{01} = yyx - w_4^{01} = xyy - w_3^{01} = xy - w_4^{01} = xyy - w_4^{01} = xyy - w_4^{01} = yxy - w_4^{01} = yy - w_4^{0$$

and, in general, we have

$$(1.2) \qquad w_{n+2}^{r_1r_2\cdots r_{n-1}r_n} = \begin{cases} w_{n+1}^{r_1r_2\cdots r_{n-1}}w_n^{r_1r_2\cdots r_{n-2}} & \text{if } r_n = 0\\ w_n^{r_1r_2\cdots r_{n-2}}w_{n+1}^{r_1r_2\cdots r_{n-1}} & \text{if } r_n = 1 \end{cases}$$

where n > 4, r_1 , r_2 , ..., $r_{n-1} \in \{0, 1\}$. We sometimes write $w_n^{r_1 r_2 \dots r_{n-2}}(x, y)$ to indicate the initial words. For simplicity we write w_n^0 (resp. w_n^1) if n > 3 and $r_1 = r_2 = \dots = r_{n-2} = 0$ (resp. 1). We sometimes write w_1^0 and w_2^0 for w_1 and w_2 , respectively. Note that

- (i) the superscript $r_1r_2...r_{n-2}$ indicates how the Fibonacci word $w_n^{r_1r_2...r_{n-2}}$ is obtained from x and y;
- (ii) the Fibonacci word $w_{n+1}^{r_1r_2\cdots r_{n-1}}$ is always an immediate predecessor of the Fibonacci word $w_{n+2}^{r_1r_2\cdots r_n}$;
- (iii) the same Fibonacci word may have several different labels;
- (iv) Knuth's Fibonacci sequence of words is $\{w_n^0\}$ while Higgins' and Turner's is $w_1, w_2, w_3^1, \ldots, w_n^1, \ldots$.

Define the reverse operation R by setting $R(x_1x_2...x_m) = x_m...x_2x_1$, where $x_1, \ldots, x_m \in \{x, y\}$. A word $w = x_1x_2...x_m$ is said to be symmetric if R(w) = w. For example, the words yxy and xyyx are symmetric.

Theorem 3:

- (i) If $w \in \mathscr{S}_n$, then $R(w) \in \mathscr{S}_n$. Moreover, if $n \ge 3$ and $w = w_n^{r_1 r_2 \dots r_{n-2}}$ where the r's are 0 or 1, then $R(w) = w_n^{s_1 s_2 \dots s_{n-2}}$ where $s_j = 1 r_j$, $j = 1, 2, \dots, n-2$.
- (ii) If v is an immediate predecessor of w, then R(v) is an immediate predecessor of R(w).

Proof: Suppose that the results are true for all positive integers less than *n*. Let $w = w_n^{r_1 r_2 \dots r_{n-2}}$ where $r_1, r_2, \dots, r_{n-2} \in \{0, 1\}$. If $r_{n-2} = 0$, then w = vu where

 $v = w_{n-1}^{r_1r_2...r_{n-3}} \in \mathscr{G}_{n-1}$ is an immediate predecessor of w

and

 $u = w_{n-2}^{r_1 r_2 \dots r_{n-4}} \in \mathcal{S}_{n-2}$ is an immediate predecessor of v.

Clearly R(w) = R(u)R(v). By the induction hypothesis,

$$R(u) = w_{n-2}^{s_1 s_2 \dots s_{n-4}} \in \mathscr{G}_{n-2}$$

is an immediate predecessor of

$$R(v) = w_{n-1}^{s_1 s_2 \dots s_{n-3}} \in \mathscr{S}_{n-1}$$

where $s_j = 1 - r_j$, j = 1, 2, ..., n - 3. Hence, R(v) is an immediate predecessor of R(w) and

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$$\begin{split} R(w) &= R(u)R(v) = w_{n_2}^{s_1s_2\cdots s_{n-4}} w_{n_1}^{s_1s_2\cdots s_{n-3}} \\ &= w_n^{s_1s_2\cdots s_{n-3}s_{n-2}} \in \mathcal{S}_n, \end{split}$$

where $s_{n-2} = 1$. The case $r_{n-2} = 1$ is proved similarly.

2. Factorization of w_n^0 into a Product of Symmetric Factors

Let
$$v_5 = y$$
, $u_5 = xyyx$, $v_6 = yxy$, $u_6 = yxyxy$. For $n \ge 7$, put
 $u_n = R(c_n)v_{n-1}c_n$,
 $v_n = v_{n-2}u_{n-2}v_{n-2}$,

where c_n equals xy if n is even and equals yx if n is odd. We sometimes write $u_n(x, y)$ and $v_n(x, y)$ for u_n and v_n , respectively. Plainly, all u_n 's and v_n 's are symmetric.

Theorem 4: For $n \ge 5$, we have

(i)	$w_n^0 = v_n u_n;$	(ii)	$v_n c_{n-1} = w_{n-1}^0;$
(iii)	$u_n = c_{n-1} w_{n-2}^0;$	(iv)	$w_n^1 = u_n v_n .$

Proof: Clearly the results are true for n = 5 and 6. Suppose n > 6 and that the results hold for all integers less than n. Then

$$\begin{aligned} v_n c_{n-1} &= v_{n-2} u_{n-2} v_{n-2} c_{n-1} &= w_{n-2}^0 w_{n-3}^0 &= w_{n-1}^0; \\ u_n &= R(c_n) (v_{n-1} c_n) &= c_{n-1} w_{n-2}^0; \\ w_n^0 &= w_{n-1}^0 w_{n-2}^0 &= v_n c_{n-1} w_{n-2}^0 &= v_n u_n. \end{aligned}$$

This proves (i)-(iii). Assertion (iv) is a consequence of Theorem 3 and the fact that u_n and v_n are symmetric.

Let w be a word in the alphabet \mathscr{A} . Designate the length of w by $\ell(w)$. In the following lemma we compute the length of the words w_n^0 , u_n , and v_n .

Lemma 5: For $n \ge 3$, we have

(i) $\ell(w_n^0) = \ell(w_{n-1}^0) + \ell(w_{n-2}^0);$ (ii) $\ell(w_n^0) = F_{n-2}\ell(x) + F_{n-1}\ell(y) = \sum_{j=1}^{n-2}\ell(w_j^0(x, y)) + \ell(y).$

For $n \ge 5$, we have

(iii)
$$l(u_n(x, y)) = l(w_{n-2}^0) + l(x) + l(y) = (F_{n-4} + 1)l(x) + (F_{n-3} + 1)l(y);$$

(iv) $l(v_n(x, y)) = l(w_{n-1}^0) - l(x) - l(y) = (F_{n-3} - 1)l(x) + (F_{n-2} - 1)l(y)$.

Proof: Both (i) and (ii) are proved by induction on n and both (iii) and (iv) follow from (i), (ii), and Theorem 4.

3. Cyclic Shift on Fibonacci Words

In this section, our main result states that every n^{th} Fibonacci word is a cyclic shift of every other n^{th} Fibonacci word. A cyclic shift operation T_n acting on words in the alphabet \mathscr{A} that have lengths n is given by

$$T_n(c_1c_2\ldots c_n) = c_2c_3\ldots c_nc_1,$$

where $c_1, c_2, \ldots, c_n \in \mathcal{A}$.

Theorem 6: Every Fibonacci word in \mathcal{S}_n is a cyclic shift of w_n^0 . More precisely, for $n \ge 3$, we have

$$w_n^{r_1r_2\cdots r_{n-2}} = \mathbb{T}^k(w_n^0),$$

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where $T = T_{I(w^0)}$ and

$$k = \sum_{j=1}^{n-2} \ell(w_{j+1}^0) r_j = \sum_{j=1}^{n-2} (F_{j-1} \ell(x) + F_j \ell(y)) r_j.$$

Proof: The result is trivial for n = 3. Suppose that n > 3 and the result is true for all integers less than n and greater than or equal to 3.

$$\begin{aligned} r_1 &= 0: \\ w_n^{r_1 r_2 \dots r_{n-2}}(x, y) &= w_{n-1}^{r_2 r_3 \dots r_{n-2}}(y, yx) \\ &= T^{k_1}(w_{n-1}^0(y, yx)) = T^k(w_n^0(x, yx)) \end{aligned}$$

where

$$k_1 = \sum_{j=2}^{n-2} \ell(w_j^0(y, yx)) r_j = \sum_{j=2}^{n-2} \ell(w_{j+1}^0(x, y)) r_j = \sum_{j=1}^{n-2} \ell(w_{j+1}^0(x, y)) r_j = k.$$

y))

$$r_{1} = 1;$$
(3.1) $w_{n}^{r_{1}r_{2}...r_{n-2}}(x, y) = w_{n-1}^{r_{2}r_{3}...r_{n-2}}(y, xy)$

$$= T^{k_1}(\omega_{n-1}^0(y, xy)) = T^{k_1}(\omega_n^{10\cdots 0}(x, y))$$

where

$$k_1 = \sum_{j=2}^{n-2} \ell(w_j^0(y, xy)) r_j = \sum_{j=2}^{n-2} \ell(w_{j+1}^{10\dots0}(x, y)) r_j = \sum_{j=2}^{n-2} \ell(w_{j+1}^0(x, y)) r_j.$$

By Theorem 4 and (iv) of Lemma 5, we have

$$\begin{split} \omega_n^1(x, y) &= u_n(x, y)v_n(x, y) = T^{\ell(v_n)}(\omega_n^0(x, y)) \\ &= T^{\ell(\omega_{n-1}^0(x, y)) - \ell(x) - \ell(y)}(\omega_n^0(x, y)). \end{split}$$

With $r_1 = r_2 = \cdots = r_{n-2} = 1$ in (1), we have $w_n^1(x, y) = T^{k_2}(w_n^{10\cdots 0}(x, y)),$

where

$$k_2 = \sum_{j=2}^{n-2} l(w_{j+1}^0(x, y)),$$

so that

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$$u^{0}\cdots u(x, y) = T^{k_3}(w_n^0(x, y)),$$

where

$$(3.2) k_3 = -k_2 + \ell(w_{n-1}^0(x, y)) - \ell(x) - \ell(y)$$
$$= -\sum_{j=1}^{n-2} \ell(w_j^0(x, y)) = -\ell(w_n^0(x, y)) + \ell(y)$$
$$= -\ell(w_n^0(x, y)) + \ell(w_2^0(x, y)),$$

in view of (ii) of Lemma 5. Combining (3.1) and (3.2), we have the desired result.

In the case that x and y are distinct alphabets, it turns out that \mathscr{G}_n consists of all the cyclic shifts of w_n^0 .

Theorem 7: Let x and y be distinct alphabets a and b, respectively. Then a word w in the alphabet $\{a, b\}$ is an n^{th} Fibonacci word if and only if w is a cyclic shift of w_n^0 .

Proof: The "only if" part is contained in Theorem 6. The "if" part is a consequence of the following Lemma (about Fibonacci numbers) whose proof is easy and is therefore omitted.

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Lemma 8: Let $n \ge 3$. For all $0 \le r \le F_n - 1$, the equation

$$\sum_{j=1}^{n-2} F_{j+1} r_j = r$$

has at least one solution $r_1, r_2, \ldots, r_{n-2}$ in $\{0, 1\}$.

We remark that this lemma also leads to the known representation theorem which states that every positive integer can be represented as a sum of a finite number of Fibonacci numbers in which each Fibonacci number occurs at most once.

4. The Case x and y Are Alphabets

As in the last theorem of Section 3, let x and y be distinct alphabets a and b, respectively. Let $q_n = w_n^{10101\cdots}$ $(n = 1, 2, \ldots)$. In this section, we locate the a's in q_n and show that all the shifts of q_n (resp. w_n^0) are distinct and hence that \mathcal{S}_n consists of precisely F_n Fibonacci words. The main result is based on the following two lemmas.

Lemma 9: Let $n \ge 3$. Then jF_{n-1} (resp. jF_{n-2}), $0 \le j \le F_n - 1$, is a complete residue system modulo F_n .

Lemma 10: (a) Let n be an odd integer greater than 4.

(i) For $1 \le j \le F_{n-4}$, let k be the unique number such that

(4.1) $1 \le k \le F_{n-2}$ and $jF_{n-3} \equiv k \pmod{F_{n-2}}$.

Then there exists a unique r_j such that

(4.2) $1 \le r_j \le F_{n-2}$ and $k \equiv r_j F_{n-1} \pmod{F_n}$.

(ii) For $1 \le i \le F_{n-3}$, let k be the unique number such that

(4.3) $F_{n-2} + 1 \le k \le F_n$ and $iF_{n-3} \equiv k - F_{n-2} \pmod{F_{n-1}}$.

Then there exists a unique t_i such that

(4.4) $1 \le t_i \le F_{n-2}$ and $k \equiv t_i F_{n-1} \pmod{F_n}$.

Furthermore,

$$(4.5) \quad \{r_i: 1 \leq j \leq F_{n-4}\} \cup \{t_i: 1 \leq i \leq F_{n-3}\} = \{1, 2, \ldots, F_{n-2}\}.$$

(b) Let n be an even integer greater than 4.

(iii) For $1 \le j \le F_{n-3}$, let k be the unique number such that

 $1 \leq k \leq F_{n-1}$ and $jF_{n-2} \equiv k \pmod{F_{n-1}}$.

Then there exists a unique r_j such that

 $1 \leq r_j \leq F_{n-2}$ and $k \equiv r_j F_{n-2} \pmod{F_n}$.

(iv) For $1 \le i \le F_{n-4}$, let k be the unique number such that

$$F_{n-1} + 1 \le k \le F_n$$
 and $iF_{n-4} \equiv k - F_{n-1} \pmod{F_{n-2}}$.

Then there exists a unique t_i such that

 $1 \leq t_i \leq F_{n-2}$ and $k \equiv t_i F_{n-2} \pmod{F_n}$.

Furthermore,

$$\{r_j: 1 \leq j \leq F_{n-3}\} \cup \{t_i: 1 \leq i \leq F_{n-4}\} = \{1, 2, \dots, F_{n-2}\}.$$

Proof: We prove (a) only.

(i) Let j and k satisfy condition (4.1). We show that (4.2) holds. Write $k = jF_{n-3} - sF_{n-2}$ where s is an integer.

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Since

 $-F_{n-2} \leq -k < jF_{n-3} - k = sF_{n-2} \leq jF_{n-3} - 1 \leq F_{n-4}F_{n-3} - 1 = F_{n-5}F_{n-2},$ we see that $0 \leq s \leq F_{n-5}$. Thus,

$$k = jF_{n-3} - sF_{n-2} = (2j + s)F_{n-1} - (j + s)F_n \equiv rF_{n-1} \pmod{F_n},$$

where $1 \le r = 2j + s \le 2F_{n-4} + F_{n-5} = F_{n-2}$. This proves (4.2).

(ii) Let *i* and *k* satisfy condition (4.3). We show that (4.4) holds. Write (4.6) $k = iF_{n-3} + F_{n-2} - sF_{n-1} = (2i - s - 1)F_{n-1} - (i - 1)F_n$

$$\equiv tF_{n-1} \pmod{F_n}$$

where s is an integer and t = 2i - s - 1. From (4.6), we have

$$F_{n-2} + 1 \le k \le tF_{n-1} = k + (i - 1)F_n \le F_n + (i - 1)F_n$$

= $iF_n \le F_{n-3}F_n = F_{n-1}F_{n-2} - 1 < F_{n-1}F_{n-2}$,

so that $1 \leq t < F_{n-2}$. This proves (4.4).

Now we prove (4.5). It is clear that the sets

$$A = \{r_j : 1 \le j \le F_{n-4}\} \text{ and } B = \{t_i : 1 \le i \le F_{n-3}\}$$

are contained in $\{1, 2, \ldots, F_{n-2}\}$. To prove equality in (4.5), we show that A has F_{n-4} elements, B has F_{n-3} elements, and that A and B are disjoint.

(a) If $r_{j_1} = r_{j_2}$, where j_1 and j_2 lie between 1 and F_{n-4} , then

 $k_{j_1} \equiv r_{j_1} F_{n-1} = r_{j_2} F_{n-1} \equiv k_{j_2} \pmod{F_n}$.

Since both k_{j_1} and k_{j_2} lie between 1 and F_{n-2} , this implies that $k_{j_1} = k_{j_2}$ and so

 $j_1F_{n-3} \equiv j_2F_{n-3} \pmod{F_{n-2}}$.

Since F_{n-2} and F_{n-3} are relatively prime, we have $j_1 = j_2$. Hence, all the r's are distinct.

(b) A similar proof shows that all the t's are distinct.

(c) If $r_j \in A$, $t_i \in B$, and $r_j = t_i$, then $k \equiv k' \pmod{F_n}$, where $r_j F_{n-1} \equiv k$ and $t_i F_{n-1} \equiv k' \pmod{F_n}$, and both k and k' lie between 1 and F_n . Therefore, we have k = k'. But this is impossible because $k \ge F_{n-2} + 1 > k'$. Thus, A and B are disjoint.

This proves (4.5), and the proof is complete.

In part (a) of Lemma 10, two injective mappings

 $r: j \in \{1, 2, \dots, F_{n-4}\} \mapsto r_j \in \{1, 2, \dots, F_{n-2}\}$ $t: i \in \{1, 2, \dots, F_{n-3}\} \mapsto t_i \in \{1, 2, \dots, F_{n-2}\}$

are defined by (4.1) and (4.2) and by (4.3) and (4.4), respectively. The disjoint union of their ranges gives the whole of $\{1, 2, \ldots, F_{n-2}\}$. Part (b) of Lemma 10 has an analogous meaning.

Now write $q_n = a_1 a_2 \dots a_{F_n}$ where $a_j \in \{a, b\}$.

Theorem 11: Let n be a positive integer greater than 3. Let $t = F_{n-1}$ if n is odd and $t = F_{n-2}$ if n is even. Then $a_k = a$ if and only if $k \equiv jt \pmod{F_n}$ for some $1 \leq j \leq F_{n-2}$.

Proof: The results are clearly true for n < 7. Now suppose that $n \ge 7$ and n is odd. Then $q_n = q_{n-2}q_{n-1}$ where

$$q_{n-2} = a_1 a_2 \dots a_{F_{n-2}}$$
 and $q_{n-1} = a_{F_{n-2}+1} \dots a_{F_n}$.

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By the induction hypothesis, the following statements are true:

(i) For $1 \le k \le F_{n-2}$, we have

 $a_k = a$ if and only if $k \equiv jF_{n-3} \pmod{F_{n-2}}$ for some $1 \leq j \leq F_{n-4}$.

(ii) For $F_{n-2} + 1 \leq k \leq F_n$, we have

 $\alpha_k = \alpha$ if and only if $k - F_{n-2} \equiv jF_{n-3} \pmod{F_{n-1}}$ for some $1 \le j \le F_{n-3}$. The result now follows from Lemmas 9 and 10. For even *n* the proof is similar.

Let $w = c_1 c_2 \dots c_n$ where c_j equals α or b. We designate by S(w) the sum (mod n) of the indices j for which $c_j = \alpha$.

Corollary 12: Let n be a positive integer greater than 2. For odd n, let

$$s = F_{n-2}$$
 and $t = F_{n-1}$;

for even n, let

 $s = F_{n-1}$ and $t = F_{n-2}$.

Suppose that $1 \le j \le F_n - 1$ and $T^{js}q_n = c_1c_2...c_{F_n}$ where $c_k \in \{a \ b\}$ and $T = T_{F_n}$. Then

- (i) $c_k = a$ if and only if $k \equiv (j + r)t \pmod{F_n}$ for some $1 \leq r \leq F_{n-2}$.
- (ii) $S(T^{js}q_n) S(T^{(j-1)s}q_n) \equiv 1 \pmod{F_n}$, and $S(T^{js}q_n) \equiv S(q_n) + j \pmod{F_n}$.
- (iii) $T^{js}q_n$ (0 $\leq j \leq F_n 1$) are F_n distinct shifts of q_n .

(iv) $T^{j}q_{n}$ (0 $\leq j \leq F_{n}$ - 1) are F_{n} distinct shifts of q_{n} .

Proof:

(i) By Theorem 11, we have

 $\begin{array}{rcl} c_k &=& \alpha \leftrightarrow k + js \equiv rt \pmod{F_n} \text{ for some } 1 \leq r \leq F_{n-2} \\ & \leftrightarrow k \equiv (j + r)t \pmod{F_n} \text{ for some } 1 \leq r \leq F_{n-2}. \end{array}$

(ii)
$$S(T^{js}q_n) - S(T^{(j-1)s}q_n) \equiv \sum_{r=1}^{F_{n-2}} (j+r)t - \sum_{r=1}^{F_{n-2}} (j+r-1)t$$

 $\equiv F_{n-2}t \equiv 1 \pmod{F_n}.$

Statement (iii) follows from (ii); statement (iv) is a consequence of (iii) and Lemma 9.

Corollary 13: Let n, s, and t be the same as in Corollary 12.

(i) If $0 \le j \le F_{n-2}$ - 1, then $T^{js}q_n$ starts with an a.

(ii) If $F_{n-2} \leq j \leq 2F_{n-2} - 1$, then $T^{js}q_n$ starts with a ba.

(iii) If $2F_{n-2} \leq j \leq F_n - 1$, then $T^{js}q_n$ starts with a bba.

(iv) If $F_{n-2} \leq j \leq F_{n-1} - 1$, then $T^{js}q_n$ starts with a b and ends with a b. *Proof:* Write $T^{js}q_n = c_1c_2...c_{F_n}$ where $c_k \in \{a, b\}$. We shall use Lemma 1, (i) of Corollary 12, and the fact that $i \equiv iF_{n-2}t \pmod{F_n}$ where i = 1, 2, and 3.

- (i) If $0 \le j \le F_{n-2} 1$, then $c_1 = a$ because $j + 1 \le F_{n-2} \le j + F_{n-2}$.
- (ii) If $F_{n-2} \leq j \leq 2F_{n-2} 1$, then the inequalities

 $j + 1 \leq 2F_{n-2} \leq j + F_{n-2}$

imply that $c_2 = a$ and hence $c_1 = b$, according to Lemma 1.

(iii) If $2F_{n-2} \leq j \leq F_n - 1$, then the inequalities

$$j + 1 \le F_n \le 3F_{n-2} \le j + F_{n-2}$$

imply that $c_3 = a$ and $c_{F_n} = a$; hence $c_1 = c_2 = b$.

(iv) If
$$F_{n-2} \leq j \leq F_{n-1} - 1$$
, then $c_1 = b$, by (ii), and since

$$F_n - 1 \equiv -F_{n-2}t \equiv F_{n-1}t$$
 and $j + 1 \leq F_{n-1} \leq 2F_{n-2} \leq j + F_{n-2}$,

we have $c_{F_n-1} = a$, so that $c_{F_n} = b$.

Theorem 14: Let n be a positive integer. Then

$$\begin{aligned} |\mathcal{G}_n| &= F_n; \quad |\mathcal{G}_n(a, b)| &= F_{n-2} = |\mathcal{G}_n(b, a)|; \\ |\mathcal{G}_n(b, b)| &= F_{n-3}; \quad |\mathcal{G}_n(b, \cdot)| &= |\mathcal{G}_n(\cdot, b)| = F_{n-1} \end{aligned}$$

Proof: The results follow from Theorem 7 and Corollaries 12 and 13.

5. Two Algorithms

In this section, the initial words are again taken to be alphabets α and b. Two algorithms will be given. Algorithm A constructs the Fibonacci word for which the multiplications involved are preassigned by means of a finite binary sequence as in (3.1) and (3.2). Algorithm B tests whether a given word in the alphabet α and b is a Fibonacci word or not.

For simplicity, we replace a by 1 and b by 0 in both algorithms so that Fibonacci words are represented by binary sequences.

Since

$$w = w_n^{r_1 r_2 \dots r_{n-2}} = T^{k_1}(w_n^0),$$

where

and

$$k_{1} = \sum_{i=1}^{n} F_{i+1} P_{i},$$

$$q_{n} = \begin{cases} T^{F_{n}-1}(w_{n}^{0}) & \text{if } n \text{ is odd} \\ T^{F_{n-1}-1}(w_{n}^{0}) & \text{if } n \text{ is even,} \end{cases}$$

it follows that $w = T^{js}q_n$ where

$$s = \begin{cases} F_{n-2} & \text{if } n \text{ is odd} \\ \\ F_{n-1} & \text{if } n \text{ is even,} \end{cases}$$
$$j \equiv \begin{cases} kF_{n-1} & \text{if } n \text{ is odd} \\ \\ kF_{n-1} - 1 & \text{if } n \text{ is even,} \end{cases}$$

and $k = k_1 + 1$. Thus, the positions of the l's in w can be determined by Corollary 12.

Algorithm A: Input a positive integer n and a binary sequence $r_1, r_2, \ldots, r_{n-2}$. This algorithm constructs the Fibonacci word $w = w_n^{r_1 r_2 \cdots r_{n-2}}$.

1) Compute
$$t = \begin{cases} F_{n-1} & \text{if } n \text{ is odd} \\ F_{n-2} & \text{if } n \text{ is even,} \end{cases}$$
 $k = \sum_{i=1}^{n-2} F_{i+1}r_i + 1,$

and j satisfying

$$j \equiv \begin{cases} kF_{n-1} & \text{if } n \text{ is odd} \\ kF_{n-1} - 1 & \text{mod } F_n \end{pmatrix}$$

$$1 \leq j \leq F_n.$$

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and

2) For $r = 1, 2, ..., F_{n-2}$, let $c_m = 1$ if $m \equiv (j + r)t \pmod{F_n}$ and $1 \le m$ $\leq F_n$; let $c_m = 0$ otherwise.

3) $w = c_1, c_2, \ldots, c_{F_n}$.

We now turn to the identification of Fibonacci words. First, observe that $n = [(\ln(\sqrt{5}(F_n + 1/2)))/\ln(\alpha)], \text{ where } \alpha = (1 + \sqrt{5})/2.$

Algorithm B: Input a positive integer h and a binary sequence $w = c_1, c_2, \ldots$, c_h . This algorithm tests whether or not w is a Fibonacci word.

- 1) Let $n = [(\ln(\sqrt{5}(h + 1/2)))/\ln(\alpha)].$
- 2) If $h \neq F_n$, then $w \notin \mathcal{S}$.
- 3) If $h = F_n$, let
 - $t = \begin{cases} F_{n-1} & \text{if } n \text{ is odd} \\ F_{n-2} & \text{if } n \text{ is even.} \end{cases}$

4) Compute the sum ${\cal S}$ of all indices i such that c_i = 1 and count the number m of 1's in w.

- 5) If $m \neq F_{n-2}$, then $w \notin \mathscr{S}$. 6) If $m = F_{n-2}$, let j be such that $1 \leq j \leq F_n$ and
 - $j \equiv S F_{n-2}(F_{n-2} + 1)t/2 \pmod{F_n}$.

7) For $r = 1, 2, ..., F_{n-2}$, let k be such that $1 \le k \le F_n$ and $k \equiv (j + r)t$ (mod F_n). If $c_k \ne 1$ for some r, then $w \notin \mathcal{S}$; otherwise $w \in \mathcal{S}$.

Note that in step 6,
$$j \equiv S(w) - S(q_n) \pmod{F_n}$$
 and so either

$$\omega = T^{js}(q_n) \in \mathcal{S}, \text{ where } s = \begin{cases} F_{n-2} & \text{if } n \text{ is odd} \\ F_{n-1} & \text{if } n \text{ is even,} \end{cases}$$
(the latter case in step 7)

 $w \notin \mathscr{S}$ or (the former case in step 7).

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