## FIBONACCI WORDS

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## 1. Introduction

In this paper, the notion of Fibonacci word is introduced and the structure of these words is investigated.

Let $\mathscr{A}$ be a nonempty set and let $x$ and $y$ be two words in the alphabet $\mathscr{A}$. A Fibonacci sequence of words derived from $x$ and $y$ is a sequence of words $w_{1}, w_{2}$, $\omega_{3}$, ... with the property that

$$
w_{1}=x, w_{2}=y, w_{n+1}=w_{n} w_{n-1} \text { or } w_{n-1} w_{n}
$$

The pair $x$ and $y$ are called the initial words of the Fibonacci sequence of words; the words $w_{n} w_{n-1}$ and $w_{n-1} w_{n}$ are called the immediate successors of $w_{n}$ and $w_{n}$ is their immediate predecessor. We remark that the Fibonacci sequence of words considered by Knuth [3, p. 85] is the one obtained as above by letting

$$
w_{n+1}=w_{n} w_{n-1} \text { for a11 } n \geq 3
$$

and the one considered by Higgins [1] and Turner [4, 5] is obtained by letting
$w_{n+1}=w_{n-1} w_{n}$ for all $n \geq 3$.
Let $\tilde{\mathscr{S}}$ be the set of all such sequences of words derived from $x$ and $y$; let $\mathscr{S}_{n}$ be the collection of words which happen to be the $n^{\text {th }}$ term of some members of $\tilde{\mathscr{S}}$. For example,

$$
\left.\mathscr{S}_{1}=\{x\}, \mathscr{S}_{2}=\{y\}, \mathscr{H}_{3}=\{y x, x y\}, \mathscr{L}_{4}=\{y x y, y y x, x y\}\right\}
$$

Denote the union of $\mathscr{S}_{n}(n=1,2, \ldots)$ by $\mathscr{S}$. Members of $\mathscr{S}_{n}$ (resp. $\mathscr{S}$ ) are called the $n$th Fibonacci words (resp. Fibonacci words). Note that each word has an obvious representation in terms of $x$ and $y$. Throughout this paper, we consider only such a representation.
Lemma 1: Let $w$ be a Fibonacci word. Then the following statements are true.
(i) If $w$ starts (resp. ends) with an $x$, then $w$ cannot end (resp. start) with an $x$.
(ii) If $w$ starts (resp. ends) with a $y$, then $w$ cannot end (resp. start) with a $y y$.
(iii) There cannot be three or more consecutive occurrences of $y$ and there cannot be two or more consecutive occurrences of $x$ in $w$.
Proof: The result is proved by mathematical induction.
Let $\mathscr{S}_{n}(x, \cdot)$ [resp. $\left.\mathscr{S}_{n}(\cdot, y)\right]$ denote those $n^{\text {th }}$ Fibonacci words which start with an $x$ (resp. end with a $y$ ) and let

$$
\mathscr{S}_{n}(x, y)=\mathscr{S}_{n}(x, \cdot) \cap \mathscr{S}_{n}(\cdot, y)
$$

Define $\mathscr{S}(x, \cdot), \mathscr{S}(\cdot, y)$, etc., in a similar way.

## Corollary 2:

$$
\begin{aligned}
\mathscr{S}_{n} & =\mathscr{S}_{n}(x, y) \cup \mathscr{S}_{n}(y, y) \cup \mathscr{S}_{n}(y, x) \quad \text { for all } n ; \\
\mathscr{S} & =\mathscr{S}(x, y) \cup \mathscr{S}(y, y) \cup \mathscr{S}(y, x)
\end{aligned}
$$

Using finite binary sequences, let us label the Fibonacci words as follows:

and, in general, we have

$$
w_{n+2}^{r_{1} r_{2} \ldots r_{n-1} r_{n}}= \begin{cases}w_{n+1}^{r_{1} r_{2} \ldots r_{n-1}} w_{n}^{r_{1} r_{2} \ldots r_{n-2}} & \text { if } r_{n}=0  \tag{1.2}\\ w_{n}^{r_{1} r_{2} \ldots r_{n-2}} w_{n+1}^{r_{1} r_{2} \ldots r_{n-1}} & \text { if } r_{n}=1\end{cases}
$$

 indicate the initial words. For simplicity we write $w_{n}^{0}$ (resp. $w_{n}^{1}$ ) if $n>3$ and $r_{1}=r_{2}=\ldots=r_{n-2}=0$ (resp. 1). We sometimes write $\omega_{1}^{0}$ and $w_{2}^{0}$ for $w_{1}$ and $w_{2}$, respectively. Note that
(i) the superscript $r_{1} r_{2} \ldots r_{n-2}$ indicates how the Fibonacci word $w_{n}^{r_{1} r_{2} \ldots r_{n-2}}$ is obtained from $x$ and $y$;
(ii) the Fibonacci word $w_{n+1}^{r_{1} r_{2} \cdots r_{n-1}}$ is always an immediate predecessor of the Fibonacci word $w_{n+2}^{r_{1} r_{2} \ldots r_{n}}$;
(iii) the same Fibonacci word may have several different labels;
(iv) Knuth's Fibonacci sequence of words is $\left\{w_{n}^{0}\right\}$ while Higgins' and Turner's is $w_{1}, w_{2}, w_{3}^{1}, \ldots, w_{n}^{1}, \ldots$.
Define the reverse operation $R$ by setting $R\left(x_{1} x_{2} \ldots x_{m}\right)=x_{m} \ldots x_{2} x_{1}$, where $x_{1}, \ldots, x_{m} \in\{x, y\}$. A word $w=x_{1} x_{2} \ldots x_{m}$ is said to be symmetric if $R(w)=w$. For example, the words $y x y$ and $x y y x$ are symmetric.

Theorem 3:
(i) If $w \in \mathscr{S}_{n}$, then $R(w) \in \mathscr{S}_{n}$. Moreover, if $n \geq 3$ and $w=w_{n}^{r_{1} r_{2} \ldots r_{n-2}}$ where the $r^{\prime}$ s are 0 or 1 , then $R(w)=w_{n}^{s_{1} s_{2} \ldots s_{n-2}}$ where $s_{j}=1-r_{j}, j=1,2$, ..., $n-2$.
(ii) If $v$ is an immediate predecessor of $w$, then $R(v)$ is an immediate predecessor of $R(w)$.
Proof: Suppose that the results are true for all positive integers less than $n$. Let $w=w_{n}^{r_{1} r_{2} \ldots r_{n-2}}$ where $r_{1}, r_{2}, \ldots, r_{n-2} \in\{0,1\}$. If $r_{n-2}=0$, then $w=v u$ where
$v=w_{n-1}^{r_{1} r_{2} \ldots r_{n-3}} \in \mathscr{S}_{n-1}$ is an immediate predecessor of $w$ $u=\omega_{n-2}^{r_{1} r_{2} \ldots r_{n-4} \in \mathscr{S}_{n-2} \text { is an immediate predecessor of } v . ~ . ~ . ~}$
Clearly $R(w)=R(u) R(v)$. By the induction hypothesis,

$$
R(\mathcal{u})=w_{n-2}^{s_{1} s_{2} \cdots s_{n-4} \in \mathscr{S}_{n-2}}
$$

is an immediate predecessor of

$$
R(v)=w_{n-1}^{s_{1} s_{2} \cdots s_{n-3} \in \mathscr{S}_{n-1}, ~}
$$

where $s_{j}=1-r_{j}, j=1,2, \ldots, n-3$. Hence, $R(v)$ is an immediate predecessor of $R(w)$ and

$$
\begin{aligned}
R(w)=R(u) R(v) & =w_{n-2}^{s_{1}} s_{2} \cdots s_{n-4} \omega_{n-1}^{s_{1} s_{2}} \ldots s_{n-3} \\
& =w_{n}^{s_{1} s_{2} \ldots s_{n-3} s_{n-2} \in \mathscr{S}_{n},}
\end{aligned}
$$

where $s_{n-2}=1$. The case $r_{n-2}=1$ is proved similarly.

## 2. Factorization of $w_{n}^{0}$ into a Product of Symmetric Factors

Let $v_{5}=y, u_{5}=x y y x, v_{6}=y x y, u_{6}=y x y x y$. For $n \geq 7$, put

$$
\begin{aligned}
& u_{n}=R\left(c_{n}\right) v_{n-1} c_{n} \\
& v_{n}=v_{n-2} u_{n-2} v_{n-2}
\end{aligned}
$$

where $c_{n}$ equals $x y$ if $n$ is even and equals $y x$ if $n$ is odd. We sometimes write $u_{n}(x, y)$ and $v_{n}(x, y)$ for $u_{n}$ and $v_{n}$, respectively. Plainly, all $u_{n}^{\prime} s$ and $v_{n}^{\prime} s$ are symmetric.

Theorem 4: For $n \geq 5$, we have
(i) $w_{n}^{0}=v_{n} u_{n}$;
(ii) $v_{n} c_{n-1}=w_{n-1}^{0}$;
(iii) $u_{n}=c_{n-1} w_{n-2}^{0}$;
(iv) $w_{n}^{1} \stackrel{n}{=} u_{n} v_{n}$.

Proof: Clearly the results are true for $n=5$ and 6 . Suppose $n>6$ and that the results hold for all integers less than $n$. Then

$$
\begin{aligned}
v_{n} c_{n-1} & =v_{n-2} u_{n-2} v_{n-2} c_{n-1}=w_{n-2}^{0} w_{n-3}^{0}=w_{n-1}^{0} ; \\
u_{n} & =R\left(c_{n}\right)\left(v_{n-1} c_{n}\right)=c_{n-1} w_{n-2}^{0} ; \\
w_{n}^{0} & =w_{n-1}^{0} w_{n-2}^{0}=v_{n} c_{n-1} w_{n-2}^{0}=v_{n} u_{n} .
\end{aligned}
$$

This proves (i)-(iii). Assertion (iv) is a consequence of Theorem 3 and the fact that $u_{n}$ and $v_{n}$ are symmetric.

Let $w$ be a word in the alphabet $\mathscr{A}$. Designate the length of $w$ by $l(w)$. In the following lemma we compute the length of the words $w_{n}^{0}, u_{n}$, and $v_{n}$.
Lemma 5: For $n \geq 3$, we have

$$
\begin{align*}
& l\left(w_{n}^{0}\right)=l\left(w_{n-1}^{0}\right)+l\left(w_{n-2}^{0}\right) ;  \tag{i}\\
& l\left(w_{n}^{0}\right)=F_{n-2} l(x)+F_{n-1} l(y)=\sum_{j=1}^{n-2} l\left(w_{j}^{0}(x, y)\right)+l(y) .
\end{align*}
$$

For $n \geq 5$, we have

$$
\begin{align*}
& l\left(u_{n}(x, y)\right)=l\left(w_{n-2}^{0}\right)+l(x)+l(y)=\left(F_{n-4}+1\right) l(x)+\left(F_{n-3}+1\right) l(y) ;  \tag{iii}\\
& l\left(v_{n}(x, y)\right)=l\left(w_{n-1}^{0}\right)-l(x)-l(y)=\left(F_{n-3}-1\right) l(x)+\left(F_{n-2}-1\right) l(y) . \tag{iv}
\end{align*}
$$

Proof: Both (i) and (ii) are proved by induction on $n$ and both (iii) and (iv) follow from (i), (ii), and Theorem 4.

## 3. Cyclic Shift on Fibonacci Words

In this section, our main result states that every $n^{\text {th }}$ Fibonacci word is a cyclic shift of every other $n^{\text {th }}$ Fibonacci word. A cyclic shift operation $T_{n}$ acting on words in the alphabet $\mathscr{A}$ that have lengths $n$ is given by

$$
T_{n}\left(c_{1} c_{2} \ldots c_{n}\right)=c_{2} c_{3} \ldots c_{n} c_{1}
$$

where $c_{1}, c_{2}, \ldots, c_{n} \in \mathscr{A}$.
Theorem 6: Every Fibonacci word in $\mathscr{S}_{n}$ is a cyclic shift of $\omega_{n}^{0}$. More precise$1 y$, for $n \geq 3$, we have

$$
w_{n}^{r_{1} r_{2} \cdots r_{n-2}}=T^{k}\left(w_{n}^{0}\right),
$$

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where $T=T_{l\left(\omega_{n}^{0}\right)}$ and

$$
k=\sum_{j=1}^{n-2} l\left(w_{j+1}^{0}\right) r_{j}=\sum_{j=1}^{n-2}\left(F_{j-1} l(x)+F_{j} l(y)\right) r_{j}
$$

Proof: The result is trivial for $n=3$. Suppose that $n>3$ and the result is true for all integers less than $n$ and greater than or equal to 3 .

$$
r_{1}=0:
$$

$$
\begin{aligned}
\omega_{n}^{r_{1} r_{2} \ldots r_{n-2}}(x, y) & =w_{n-1}^{r_{2} r_{3} \ldots r_{n-2}}(y, y x) \\
& =T^{k_{1}}\left(w_{n-1}^{0}(y, y x)\right)=T^{k}\left(w_{n}^{0}(x, y)\right)
\end{aligned}
$$

where

$$
k_{1}=\sum_{j=2}^{n-2} l\left(w_{j}^{0}(y, y x)\right) r_{j}=\sum_{j=2}^{n-2} l\left(w_{j+1}^{0}(x, y)\right) r_{j}=\sum_{j=1}^{n-2} l\left(w_{j+1}^{0}(x, y)\right) r_{j}=k .
$$

$$
r_{1}=1:
$$

$$
\begin{align*}
w_{n}^{r_{1} r_{2} \cdots r_{n-2}}(x, y) & =w_{n-1}^{r_{2} r_{3} \cdots r_{n-2}}(y, x y)  \tag{3.1}\\
& =T^{k_{1}}\left(w_{n-1}^{0}(y, x y)\right)=T^{k_{1}}\left(w_{n}^{10} \cdots{ }^{0}(x, y)\right)
\end{align*}
$$

where

$$
k_{1}=\sum_{j=2}^{n-2} l\left(w_{j}^{0}(y, x y)\right) r_{j}=\sum_{j=2}^{n-2} l\left(w_{j+1}^{10 \ldots 0}(x, y)\right) r_{j}=\sum_{j=2}^{n-2} l\left(w_{j+1}^{0}(x, y)\right) r_{j}
$$

By Theorem 4 and (iv) of Lemma 5, we have

$$
\begin{aligned}
& w_{n}^{1}(x, y)=u_{n}(x, y) v_{n}(x, y)=T l\left(v_{n}\right)\left(w_{n}^{0}(x, y)\right) \\
&=T l\left(w_{n-1}^{0}(x, y)\right)-l(x)-\ell(y) \\
&\left(w_{n}^{0}(x, y)\right)
\end{aligned}
$$

With $r_{1}=r_{2}=\cdots=r_{n-2}=1$ in (1), we have

$$
w_{n}^{1}(x, y)=T^{k_{2}}\left(w_{n}^{10 \cdots 0}(x, y)\right)
$$

where
so that:

$$
k_{2}=\sum_{j=2}^{n-2} l\left(w_{j+1}^{0}(x, y)\right)
$$

$$
w^{10 \cdots 0}(x, y)=T^{k_{3}}\left(w_{n}^{0}(x, y)\right)
$$

where

$$
\begin{align*}
k_{3} & =-k_{2}+l\left(w_{n-1}^{0}(x, y)\right)-l(x)-l(y)  \tag{3.2}\\
& =-\sum_{j=1}^{n-2} l\left(w_{j}^{0}(x, y)\right)=-l\left(w_{n}^{0}(x, y)\right)+l(y) \\
& =-l\left(w_{n}^{0}(x, y)\right)+l\left(w_{2}^{0}(x, y)\right),
\end{align*}
$$

in view of (ii) of Lemma 5. Combining (3.1) and (3.2), we have the desired result.

In the case that $x$ and $y$ are distinct alphabets, it turns out that $\mathscr{S}_{n}$ consists of all the cyclic shifts of $w_{n}^{0}$.
Theorem 7: Let $x$ and $y$ be distinct alphabets $a$ and $b$, respectively. Then a word $w$ in the alphabet $\{a, b\}$ is an $n^{\text {th }}$ Fibonacci word if and only if $w$ is a cyclic shift of $\omega_{n}^{0}$.
Proof: The "only if" part is contained in Theorem 6. The "if" part is a consequence of the following Lemma (about Fibonacci numbers) whose proof is easy and is therefore omitted.

Lemma 8: Let $n \geq 3$. For all $0 \leq r \leq F_{n}-1$, the equation

$$
\sum_{j=1}^{n-2} F_{j+1} r_{j}=r
$$

has at least one solution $r_{1}, r_{2}, \ldots, r_{n-2}$ in $\{0,1\}$.
We remark that this lemma also leads to the known representation theorem which states that every positive integer can be represented as a sum of a finite number of Fibonacci numbers in which each Fibonacci number occurs at most once.

## 4. The Case $x$ and $y$ Are Alphabets

As in the last theorem of Section 3, let $x$ and $y$ be distinct alphabets $a$ and $b$, respectively. Let $q_{n}=w_{n}^{l 010 l \ldots} \quad(n=1,2, \ldots)$. In this section, we locate the $\alpha^{\prime}$ s in $q_{n}$ and show that all the shifts of $q_{n}$ (resp. $w_{n}^{0}$ ) are distinct and hence that $\mathscr{I}_{n}$ consists of precisely $F_{n}$ Fibonacci words. The main result is based on the following two lemmas.
Lemma 9: Let $n \geq 3$. Then $j F_{n-1}$ (resp. $j F_{n-2}$ ), $0 \leq j \leq F_{n}-1$, is a complete residue system modulo $F_{n}$.

Lemma 10: (a) Let $n$ be an odd integer greater than 4.
(i) For $1 \leq j \leq F_{n-4}$, let $k$ be the unique number such that
(4.1) $1 \leq k \leq F_{n-2}$ and $j F_{n-3} \equiv k\left(\bmod F_{n-2}\right)$.

Then there exists a unique $r_{j}$ such that

$$
\begin{equation*}
1 \leq r_{j} \leq F_{n-2} \text { and } k \equiv r_{j} F_{n-1}\left(\bmod F_{n}\right) \tag{4.2}
\end{equation*}
$$

(ii) For $1 \leq i \leq F_{n-3}$, let $k$ be the unique number such that

$$
\begin{equation*}
F_{n-2}+1 \leq k \leq F_{n} \quad \text { and } \quad i F_{n-3} \equiv k-F_{n-2}\left(\bmod F_{n-1}\right) \tag{4.3}
\end{equation*}
$$

Then there exists a unique $t_{i}$ such that
(4.4) $1 \leq t_{i} \leq F_{n-2}$ and $k \equiv t_{i} F_{n-1}\left(\bmod F_{n}\right)$.

Furthermore,
(4.5) $\quad\left\{r_{j}: 1 \leq j \leq F_{n-4}\right\} \cup\left\{t_{i}: 1 \leq i \leq F_{n-3}\right\}=\left\{1,2, \ldots, F_{n-2}\right\}$.
(b) Let $n$ be an even integer greater than 4.
(iii) For $1 \leq j \leq F_{n-3}$, let $k$ be the unique number such that

$$
1 \leq k \leq F_{n-1} \quad \text { and } \quad j F_{n-2} \equiv k\left(\bmod F_{n-1}\right)
$$

Then there exists a unique $r_{j}$ such that

$$
1 \leq r_{j} \leq F_{n-2} \text { and } k \equiv r_{j} F_{n-2}\left(\bmod F_{n}\right)
$$

(iv) For $1 \leq i \leq F_{n-4}$, let $k$ be the unique number such that

$$
F_{n-1}+1 \leq k \leq F_{n} \quad \text { and } \quad i F_{n-4} \equiv k-F_{n-1}\left(\bmod F_{n-2}\right)
$$

Then there exists a unique $t_{i}$ such that

$$
1 \leq t_{i} \leq F_{n-2} \text { and } k \equiv t_{i} F_{n-2} \quad\left(\bmod F_{n}\right)
$$

Furthermore,

$$
\left\{r_{j}: 1 \leq j \leq F_{n-3}\right\} \cup\left\{t_{i}: 1 \leq i \leq F_{n-4}\right\}=\left\{1,2, \ldots, F_{n-2}\right\}
$$

Proof: We prove (a) only.
(i) Let $j$ and $k$ satisfy condition (4.1). We show that (4.2) holds. Write $k=j F_{n-3}-s F_{n-2} \quad$ where $s$ is an integer.

Since

$$
-F_{n-2} \leq-k<j F_{n-3}-k=s F_{n-2} \leq j F_{n-3}-1 \leq F_{n-4} F_{n-3}-1=F_{n-5} F_{n-2}
$$

we see that $0 \leq s \leq F_{n-5}$. Thus,

$$
k=j F_{n-3}-s F_{n-2}=(2 j+s) F_{n-1}-(j+s) F_{n} \equiv r F_{n-1}\left(\bmod F_{n}\right)
$$

where $1 \leq r=2 j+s \leq 2 F_{n-4}+F_{n-5}=F_{n-2}$. This proves (4.2).
(ii) Let $i$ and $k$ satisfy condition (4.3). We show that (4.4) holds. Write

$$
\begin{align*}
k=i F_{n-3}+F_{n-2}-s F_{n-1} & =(2 i-s-1) F_{n-1}-(i-1) F_{n}  \tag{4.6}\\
& \equiv t F_{n-1}\left(\bmod F_{n}\right)
\end{align*}
$$

where $s$ is an integer and $t=2 i-s-1$. From (4.6), we have

$$
\begin{aligned}
F_{n-2}+1 \leq k \leq t F_{n-1} & =k+(i-1) F_{n} \leq F_{n}+(i-1) F_{n} \\
& =i F_{n} \leq F_{n-3} F_{n}=F_{n-1} F_{n-2}-1<F_{n-1} F_{n-2}
\end{aligned}
$$

so that $1 \leq t<F_{n-2}$. This proves (4.4).
Now we prove (4.5). It is clear that the sets

$$
A=\left\{r_{j}: 1 \leq j \leq F_{n-4}\right\} \quad \text { and } \quad B=\left\{t_{i}: 1 \leq i \leq F_{n-3}\right\}
$$

are contained in $\left\{1,2, \ldots, F_{n-2}\right\}$. To prove equality in (4.5), we show that $A$ has $F_{n-4}$ elements, $B$ has $F_{n-3}$ elements, and that $A$ and $B$ are disjoint.
(a) If $r_{j_{1}}=r_{j_{2}}$, where $j_{1}$ and $j_{2}$ lie between 1 and $F_{n-4}$, then $k_{j_{1}} \equiv r_{j_{1}} F_{n-1}=r_{j_{2}} F_{n-1} \equiv k_{j_{2}}\left(\bmod F_{n}\right)$.
Since both $k_{j_{1}}$ and $k_{j_{2}}$ lie between 1 and $F_{n-2}$, this implies that $k_{j_{1}}=k_{j_{2}}$ and so

$$
j_{1} F_{n-3} \equiv j_{2} F_{n-3}\left(\bmod F_{n-2}\right)
$$

Since $F_{n-2}$ and $F_{n-3}$ are relatively prime, we have $j_{1}=j_{2}$. Hence, all the p's are distinct.
(b) A similar proof shows that all the $t^{\prime}$ 's are distinct.
(c) If $r_{j} \in A, t_{i} \in B$, and $r_{j}=t_{i}$, then $k \equiv k^{\prime}\left(\bmod F_{n}\right)$, where $r_{j} F_{n-1} \equiv k$ and $t_{i} F_{n-1} \equiv \mathcal{K}^{\prime}\left(\bmod F_{n}\right)$, and both $k$ and $k^{\prime}$ lie between 1 and $F_{n}$. Therefore, we have $k=K^{\prime}$. But this is impossible because $k \geq F_{n-2}+1>K^{\prime}$. Thus, $A$ and $B$ are disjoint.

This proves (4.5), and the proof is complete.
In part (a) of Lemma 10, two injective mappings

$$
\begin{aligned}
& r: j \in\left\{1,2, \ldots, F_{n-4}\right\} \mapsto r_{j} \in\left\{1,2, \ldots, F_{n-2}\right\} \\
& t: i \in\left\{1,2, \ldots, F_{n-3}\right\} \mapsto t_{i} \in\left\{1,2, \ldots, F_{n-2}\right\}
\end{aligned}
$$

are defined by (4.1) and (4.2) and by (4.3) and (4.4), respectively. The disjoint union of their ranges gives the whole of $\left\{1,2, \ldots, F_{n-2}\right\}$. Part (b) of Lemma 10 has an analogous meaning.

Now write $q_{n}=a_{1} a_{2} \ldots a_{F_{n}}$ where $a_{j} \in\{a, b\}$.
Theorem 11: Let $n$ be a positive integer greater than 3 . Let $t=F_{n-1}$ if $n$ is odd and $t=F_{n-2}$ if $n$ is even. Then $\alpha_{k}=\alpha$ if and only if $k \equiv j t\left(m o d F_{n}\right)$ for some $1 \leq j \leq E_{n-2}$.
Proof: The results are clearly true for $n<7$. Now suppose that $n \geq 7$ and $n$ is odd. Then $q_{n}=q_{n-2} q_{n-1}$ where

$$
q_{n-2}=\alpha_{1} a_{2} \ldots a_{F_{n-2}} \text { and } q_{n-1}=a_{F_{n-2}+1} \ldots \alpha_{F_{n}}
$$

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By the induction hypothesis, the following statements are true:
(i) For $1 \leq k \leq F_{n-2}$, we have

$$
\begin{aligned}
& \alpha_{k}=a \text { if and only if } k \equiv j F_{n-3}\left(\bmod F_{n-2}\right) \text { for some } 1 \leq j \leq F_{n-4} \\
& \text { (ii) For } F_{n-2}+1 \leq k \leq F_{n} \text {, we have } \\
& \alpha_{k}=\alpha \text { if and only if } k-F_{n-2} \equiv j F_{n-3}\left(\bmod F_{n-1}\right) \text { for some } 1 \leq j \leq F_{n-3}
\end{aligned}
$$ The result now follows from Lemmas 9 and 10 . For even $n$ the proof is similar.

Let $w=c_{1} c_{2} \ldots c_{n}$ where $c_{j}$ equals $a$ or $b$. We designate by $S(w)$ the sum $(\bmod n)$ of the indices $j$ for which $c_{j}=a$.
Corollary 12: Let $n$ be a positive integer greater than 2 . For odd $n$, let

$$
s=F_{n-2} \quad \text { and } \quad t=F_{n-1}
$$

for even $n$, let

$$
s=F_{n-1} \quad \text { and } \quad t=F_{n-2}
$$

Suppose that $1 \leq j \leq F_{n}-1$ and $T^{j s} q_{n}=c_{1} c_{2} \ldots c_{F_{n}}$ where $c_{k} \in\{\alpha \quad b\}$ and $T=$ $T_{F_{n}}$. Then
(i) $c_{k}=a$ if and only if $k \equiv(j+r) t\left(\bmod F_{n}\right)$ for some $1 \leq r \leq F_{n-2}$.
(ii) $S\left(T^{j s} q_{n}\right)-S\left(T^{(j-1) s} q_{n}\right) \equiv 1\left(\bmod F_{n}\right)$, and $S\left(T^{j s} q_{n}\right) \equiv S\left(q_{n}\right)+j\left(\bmod F_{n}\right)$.
(iv)

Proof:
(i) By Theorem 11, we have

$$
\begin{gathered}
c_{k}=a \Leftrightarrow k+j s \equiv r t\left(\bmod F_{n}\right) \text { for some } 1 \leq r \leq F_{n-2} \\
\leftrightarrow k \equiv(j+r) t\left(\bmod F_{n}\right) \text { for some } 1 \leq r \leq F_{n-2} \\
S\left(T^{j s} q_{n}\right)-S\left(T^{(j-1) s} q_{n}\right)
\end{gathered} \begin{gathered}
\equiv \sum_{r=1}^{F_{n-2}}(j+r) t-\sum_{r=1}^{F_{n-2}}(j+r-1) t \\
\equiv F_{n-2} t \equiv 1\left(\bmod F_{n}\right)
\end{gathered}
$$

(ii)

Statement (iii) follows from (ii); statement (iv) is a consequence of (iii) and Lemma 9.
Corollary 13: Let $n, s$, and $t$ be the same as in Corollary 12.
(i) If $0 \leq j \leq F_{n-2}-1$, then $T^{j s} q_{n}$ starts with an $\alpha$.
(ii) If $F_{n-2} \leq j \leq 2 F_{n-2}-1$, then $T^{j s} q_{n}$ starts with a $b a$.
(iii) If $2 F_{n-2} \leq j \leq F_{n}-1$, then $T^{j s} q_{n}$ starts with a $b b a$.
(iv) If $F_{n-2} \leq j \leq F_{n-1}-1$, then $T^{j s} q_{n}$ starts with a $b$ and ends with a $b$.

Proof: Write $T^{j s} q_{n}=c_{1} c_{2} \ldots c_{F_{n}}$ where $c_{k} \in\{a, b\}$. We shall use Lemma 1, (i) of Corollary 12, and the fact that $i \equiv i F_{n-2} t\left(\bmod F_{n}\right)$ where $i=1$, 2, and 3 .
(i) If $0 \leq j \leq F_{n-2}-1$, then $c_{1}=\alpha$ because $j+1 \leq F_{n-2} \leq j+F_{n-2}$.
(ii) If $F_{n-2} \leq j \leq 2 F_{n-2}-1$, then the inequalities $j+1 \leq 2 F_{n-2} \leq j+F_{n-2}$
imply that $c_{2}=a$ and hence $c_{1}=b$, according to Lemma 1 .
(iii) If $2 F_{n-2} \leq j \leq F_{n}-1$, then the inequalities

$$
j+1 \leq F_{n} \leq 3 F_{n-2} \leq j+F_{n-2}
$$

imply that $c_{3}=\alpha$ and $c_{F_{n}}=\alpha$; hence $c_{1}=c_{2}=b$.

$$
\begin{aligned}
& \text { (iv) If } F_{n-2} \leq j \leq F_{n-1}-1 \text {, then } c_{1}=b \text {, by (ii), and since } \\
& F_{n}-1 \equiv-F_{n-2} t \equiv F_{n-1} t \text { and } j+1 \leq F_{n-1} \leq 2 F_{n-2} \leq j+F_{n-2},
\end{aligned}
$$

we have $c_{F_{n}-1}=a$, so that $c_{F_{n}}=b$.
Theorem 14: Let $n$ be a positive integer. Then

$$
\begin{aligned}
& \left|\mathscr{S}_{n}\right|=F_{n} ;\left|\mathscr{S}_{n}(a, b)\right|=F_{n-2}=\left|\mathscr{S}_{n}(b, a)\right| ; \\
& \left|\mathscr{S}_{n}(b, b)\right|=F_{n-3} ;\left|\mathscr{S}_{n}(b, \cdot)\right|=\left|\mathscr{S}_{n}(\cdot, b)\right|=F_{n-1} .
\end{aligned}
$$

Proof: The results follow from Theorem 7 and Corollaries 12 and 13.

## 5. Two Algorithms

In this section, the initial words are again taken to be alphabets $\alpha$ and $b$. Two algorithms will be given. Algorithm A constructs the Fibonacci word for which the multiplications involved are preassigned by means of a finite binary sequence as in (3.1) and (3.2). Algorithm B tests whether a given word in the alphabet $a$ and $b$ is a•Fibonacci word or not.

For simplicity, we replace $a$ by 1 and $b$ by 0 in both algorithms so that Fibonacci words are represented by binary sequences.

Since

$$
w=\omega_{n}^{r_{1} r_{2} \ldots r_{n-2}=T^{k_{1}}\left(\omega_{n}^{0}\right), ~ ; ~}
$$

where

$$
k_{1}=\sum_{i=1}^{n-2} F_{i+1} r_{i},
$$

$$
q_{n}= \begin{cases}T^{F_{n}-1}\left(\omega_{n}^{0}\right) & \text { if } n \text { is odd } \\ T^{F_{n-1}-1}\left(\omega_{n}^{0}\right) & \text { if } n \text { is even }\end{cases}
$$

it follows that $w=T^{j s} q_{n}$ where

$$
\begin{aligned}
& s= \begin{cases}F_{n-2} \text { if } n \text { is odd } \\
F_{n-1} \text { if } n \text { is even, }\end{cases} \\
& j \equiv \begin{cases}k F_{n-1}\left(\bmod F_{n}\right) & \text { if } n \text { is odd } \\
k F_{n-1}-1 & \text { if } n \text { is even, }\end{cases}
\end{aligned}
$$

and $k=k_{1}+1$. Thus, the positions of the 1 's in $w$ can be determined by Corollary 12.
Algorithm A: Input a positive integer $n$ and a binary sequence $r_{1}, r_{2}, \ldots$, $r_{n-2}$. This algorithm constructs the Fibonacci word $w=w_{n}^{r_{1} r_{2} \ldots r_{n-2}}$.

1) Compute $t=\left\{\begin{array}{ll}F_{n-1} & \text { if } n \text { is odd } \\ F_{n-2} & \text { if } n \text { is even, }\end{array} \quad k=\sum_{i=1}^{n-2} F_{i+1} r_{i}+1\right.$,
and $j$ satisfying

$$
j \equiv \begin{cases}k F_{n-1} & \left(\bmod F_{n}\right) \\ k F_{n-1}-1 & \text { if } n \text { is odd } \\ \text { if } n \text { is even }\end{cases}
$$

and $1 \leq j \leq F_{n}$.

```
    2) For r = 1, 2, ..., F F n-2, let cm}=1\mathrm{ if m 
\leq En; let cm
    3) w = c
    We now turn to the identification of Fibonacci words. First, observe that
n=[(ln}(\sqrt{}{5}(\mp@subsup{F}{n}{}+1/2)))/\operatorname{ln}(\alpha)],\mathrm{ where }\alpha=(1+\sqrt{}{5})/2
```

Algorithm $B$ : Input a positive integer $h$ and a binary sequence $w=c_{1}, c_{2}, \ldots$, $c_{h}$. This algorithm tests whether or not $w$ is a Fibonacci word.

1) Let $n=[(\ln (\sqrt{5}(h+1 / 2))) / \ln (\alpha)]$.
2) If $h \neq F_{n}$, then $\omega \notin \mathscr{S}$.
3) If $h=F_{n}$, let $t= \begin{cases}F_{n-1} & \text { if } n \text { is odd } \\ F_{n-2} & \text { if } n \text { is even. }\end{cases}$
4) Compute the sum $S$ of all indices $i$ such that $c_{i}=1$ and count the number $m$ of 1 's in $\omega$.
5) If $m \neq F_{n-2}$, then $w \notin \mathscr{S}$.
6) If $m=F_{n-2}$, let $j$ be such that $1 \leq j \leq F_{n}$ and $j \equiv S-F_{n-2}\left(F_{n-2}+1\right) t / 2\left(\bmod F_{n}\right)$.
7) For $r=1,2, \ldots, F_{n-2}$, let $k$ be such that $1 \leq k \leq F_{n}$ and $k \equiv(j+r) t$ $\left(\bmod F_{n}\right)$. If $c_{k} \neq 1$ for some $r$, then $w \notin \mathscr{S}$; otherwise $w \in \mathscr{S}$.

Note that in step $6, j \equiv S(w)-S\left(q_{n}\right)\left(\bmod F_{n}\right)$ and so either

$$
w=T^{j s}\left(q_{n}\right) \in \mathscr{f}, \text { where } s= \begin{cases}F_{n-2} & \text { if } n \text { is odd } \quad \text { (the latter case } \\ F_{n-1} & \text { if } n \text { is even, } \\ \text { in step 7) }\end{cases}
$$

or $\quad \omega \notin \mathscr{S}$ (the former case in step 7).

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