THE FIBONACCI AND PARACHUTE INEQUALITIES FOR ℓ_1-METRICS

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(Submitted March 1990)

1. Introduction

Let X be a finite set with $|X| = n$. A *metric* on X is a real valued function d defined on all pairs of points of X and satisfying the *triangle inequality*:

(1) $\quad d(i, j) + d(j, k) \geq d(i, k)$

for all triples (i, j, k) of points of X. We allow $d(i, j) = 0$ for some pairs (i, j); so we use the term "metric" for denoting what is usually called semi-metric. We set $d(i, j) = d(j, i)$ for all pairs (i, j) and $d(i, i) = 0$ for all points i of X. The pair (X, d) is called a *metric space*. The ℓ_1-metric on R^m is defined by:

$$d(x, y) = \|x - y\|_1 = \sum_{1 \leq i \leq m} |x_i - y_i|.$$

A metric space (X, d) is isometrically ℓ_1-embeddable if there exist points x_0, x_1, ..., x_n in some space R^m such that

$$d(i, j) = \|x_i - x_j\|_1 \quad \text{for all } 0 \leq i < j \leq n.$$

The family of all metrics d on X which are isometrically ℓ_1-embeddable forms a cone: $C(X) = C_n$, called *cut cone* (or *Hamming cone*). The cut cone C_n is generated by the *cut metrics* d_S for subsets S of X, where

$$d_S(i, j) = 1 \text{ if } |S \cap \{i, j\}| = 1 \quad \text{and} \quad d_S(i, j) = 0 \quad \text{otherwise}.$$

Therefore, a metric d on X is isometrically ℓ_1-embeddable if and only if d is the conic hull of cut metrics:

$$d = \sum_{S \subseteq X} \lambda_S d_S, \text{ with } \lambda_S \geq 0.$$

The cut metrics d_S correspond, in graph terminology, to the cuts $\delta(S)$; we shall use the latter terminology here. The study of the ℓ_1-embeddable finite metric spaces, i.e., of the cut cone C_n, was started in 1960 in [5] and continued, e.g., in [1], [3], [6], [8], [9], and [10]. If d is rational valued, then d is isometrically ℓ_1-embeddable if and only if kd is isometrically embeddable into the vertex set of the hypercube of R^m for some integers k, m ([2]).

Given a vector $v = (v_{ij})_{1 \leq i \leq j < n}$, the inequality $v.x \leq 0$ is called *valid* over the cut cone C_n if it is satisfied by all points of C_n or, in other words, by all metrics on n points which are isometrically ℓ_1-embeddable. The *roots* of inequality $v.x \leq 0$ are the cuts $\delta(S)$ satisfying equality: $v.\delta(S) = 0$. The rank of inequality $v.x \leq 0$ is the rank of its set of roots. Geometrically, valid inequalities correspond to faces of the cone C_n while valid inequalities of highest possible rank:

$$\binom{n}{2} - 1 = n(n - 1)/2 - 1$$

define facets of C_n.

Many examples of valid inequalities over the cut cone C_n are known; for example,

(a) the hypermetric inequalities ([5], [11], [8]) of the form

$$\sum_{1 \le i < j \le n} b_i b_j d(i, j) \le 0,$$

where $b_1, \ldots, b_n$ are integers satisfying

$$\sum_{1 \le i \le n} b_i = 1;$$

including triangle inequality (1) as a special case for $b = (1, 1, -1, 0, \ldots, 0)$.

(b) the *bicycle odd wheel inequality* [4], defined on $2k + 3$ points $\{0, 0', 1, 2, \ldots, 2k + 1\}$ for $k \ge 1$ by

$$d(0, 0') - \sum_{1 \le i \le 2k+1} (d(0, i) + d(0', i)) + \sum_{(i, j) \in C} d(i, j) \le 0, \tag{2}$$

where C denotes the cycle $(1, 2, \ldots, 2k + 1)$.

(c) the *parachute inequality* [8], denoted as Par_{2k+1}, defined on the $2k + 1$ points $\{0, 1, 2, \ldots, k, 1', 2', \ldots, k'\}$ by:

$$\mathrm{Par}_{2k+1} \cdot d = \sum_{(i, j) \in P} d(i, j) - \sum_{1 \le i \le k-1} (d(0, i) + d(0, i') + d(k, i') + d(k', i)) - d(k, k') \le 0, \tag{3}$$

where P is the path $(k, k - 1, \ldots, 2, 1, 1', 2', \ldots, (k - 1)', k')$.

(d) the *Fibonacci inequality* [10], denoted as Fib_{2k}, defined on the $2k$ points $\{0, 0', 1, 1', 2, 2', \ldots, (k - 1), (k - 1)'\}$ by

$$\mathrm{Fib}_{2k} \cdot d = \sum_{(i, j) \in Q} d(i, j) - \sum_{1 \le i \le k-1} (d(0, i) + d(0, i')) - \sum_{1 \le i \le k-2} (d(0', i) + d(0', i')) \le 0, \tag{4}$$

where Q is the path $(k - 1, k - 2, \ldots, 2, 1, 1', 2', \ldots, (k - 2)', (k - 1)')$. We call the above inequality (4) the Fibonacci inequality, since its number of roots is related to the Fibonacci number f_k ($f_1 = f_2 = 1$, $f_{k+2} = f_{k+1} + f_k$).

See Figures 1-3 for the graphic representation of inequalities (2) and (3) on seven points and inequality (4) on six points (a plain line indicates coefficient +1 and a dotted line indicates coefficient -1).

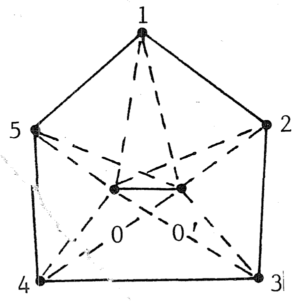

Figure 1

The Bicycle Odd Wheel Inequality on 7 Points

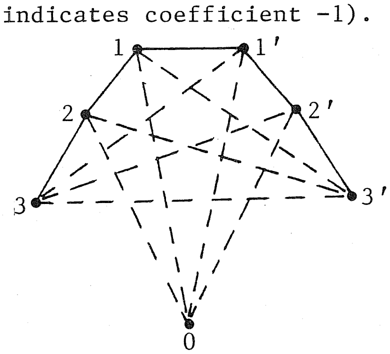

Figure 2

The Parachute Inequality Par_7 on 7 Points

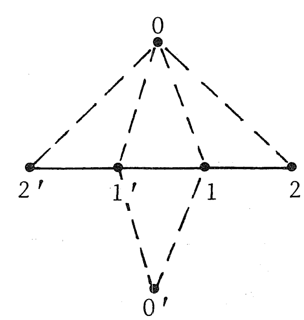

Figure 3

The Fibonacci Inequality Fib_6 on 6 Points

In this note, we consider the Fibonacci and parachute inequalities, their number of roots in terms of Fibonacci numbers, their rank, their symmetries, and the connections with the bicycle odd wheel inequality (2).

2. Parachute and Fibonacci Inequalities: Equality Case

Given a path $A = (1, 2, \ldots, n)$, a subset S of $[1, n]$ is called *alternated* along path A if

$$|S \cap \{i, i+1\}| \leq 1 \quad \text{for all } i = 1, \ldots, n-1,$$

and *pseudo-alternated* along path A if

$$|S \cap \{i, i+1\}| = 1 \quad \text{for all } i = 1, \ldots, j-1, j+1, \ldots, n-1$$

and

$$|S \cap \{j, j+1\}| = 0 \text{ or } 2 \quad \text{for some } j \in [1, n-1].$$

One observes easily that, for n even, the number of pseudo-alternated subsets S along path $A = (1, \ldots, n)$ for which nodes $1, n$ belong to S is equal to $n - 1$; an easy induction on n shows that the number of alternated subsets of $[1, n]$ along path $(1, 2, \ldots, n)$ is equal to the Fibonacci number f_{n+2} (where $[1, n]$ denotes the set of integers $1, 2, 3, \ldots, n$).

Call a cut $\delta(S)$ *symmetric* if, for $i = 1, 2, \ldots, k$, i belongs to S if and only if i' belongs to S, i.e., the involution

$$\alpha = \prod_{1 \leq i \leq k} (ii')$$

belonging to the symmetric group $\mathrm{Sym}(2k+1)$ leaves S invariant.

We describe below the roots of the parachute inequality.

Proposition 1: The roots of the parachute inequality Par_{2k+1} are the cuts $\delta(S)$ for which S is a subset of $[1, k] \cup [1', k']$ of one of the following four types:

Type 1: nodes k, k' belong to S and S is pseudo-alternated along path P.
Type 2: nodes k, k' do not belong to S and S is alternated along path Q.
Type 3: for k odd, node k belongs to S, node k' does not belong to S and (a) or (b) holds:

(a) $S = \{2', 4', \ldots, (k-1)', k\} \cup T$, where T is a subset of $\{1, 2, \ldots, k-2\}$ alternated along path $\{1, 2, \ldots, k-2\}$;
(b) $S = \{k, 1', (k-1)'\} \cup T \cup V$, where T is a subset of $\{2, 3, \ldots, k-2\}$ alternated along path $(2, 3, \ldots, k-2)$ and V is a subset of $\{2', 3', \ldots, (k-2)'\}$ such that $V \cup \{1', (k-1)'\}$ is pseudo-alternated along path $(1', 2', \ldots, (k-1)')$.

Type 3′: similar to type 3, exchanging nodes i, i' for all $i = 1, 2, \ldots, k$.

There are $2k - 1$ roots of type 1, all of them linearly independent and the only symmetric root among them is $\delta(\{1, 3, \ldots, k, 1', 3', \ldots, k'\})$ for k odd and $\delta(\{2, 4, \ldots, k, 2', 4', \ldots, k'\})$ for k even. There are f_{2k} roots of type 2, their rank is $\binom{2k-1}{2} - 2k + 3$ and there are f_k symmetric roots among them. The roots of type 3, 3′ exist only for k odd; there are altogether $2(f_k + (k-1)f_{k-1})$ such roots and there are no symmetric roots among them.

Proposition 2: (i) The number of roots [including zero, i.e., cut $\delta(\phi)$] of the parachute inequality Par_{2k+1} is equal to $f_{2k} + 2kf_{k-1} + 2f_{k-2} + 2k - 1$ for k odd and $f_{2k} + 2k - 1$ for k even, while the number of (nonzero) symmetric roots is always the Fibonacci number f_k. (ii) The parachute inequality Par_{2k+1} is facet inducing for k odd; for k even, it has rank $\binom{2k-1}{2} + 2$, but it is not valid.

Proof of Propositions 1 and 2: Given a subset S of $[1, k] \cup [1', k']$, we set

$$s = |S \cap [1, k-1]| \quad \text{and} \quad s' = |S \cap [1', (k-1)']|.$$

In order to characterize which cuts $\delta(S)$ are roots of the parachute inequality Par_{2k+1}, we distinguish four cases:

Case 1: k, $k' \in S$

Then $\delta(S)$ is a root of Par_{2k+1} if and only if $|\delta(S) \cap P| = 2k - 2$, i.e., all edges of P but one are edges of $\delta(S)$, i.e., S is pseudo-alternated along path P. So there are $2k - 1$ such roots, among them only one symmetric root:

$$\delta(\{k, \ldots, 3, 1, 1', 3', \ldots, k'\}) \quad \text{for } k \text{ odd}$$

and

$$\delta(\{k, \ldots, 4, 2, 2', 4', \ldots, k'\}) \text{ for } k \text{ even.}$$

Case 2: k, $k' \notin S$

Then $\delta(S)$ is a root of Par_{2k+1} if and only if $|\delta(S) \cap P| = 2(s + s') = 2|S|$, i.e., S is alternated along the path $(k - 1, \ldots, 1, 1', \ldots, (k - 1)')$ on $2k - 2$ nodes, so there are f_{2k} such roots. Among them, the number of symmetric roots (including zero) is equal to the number of alternated subsets along path $(2, 3, \ldots, k - 1)$, i.e., to f_k.

Case 3: $k \in S$, $k' \notin S$

Then $\delta(S)$ is a root of Par_{2k+1} if and only if $|\delta(S) \cap P| = k + 2s$. Since

$$|\delta(S) \cap P| = |\delta(S) \cap \text{Path}(k, \ldots, 1, 1')| + |\delta(S) \cap \text{Path}(1', \ldots, k')|,$$

with the first term being less than or equal to $2s + 2$, we have to distinguish two cases.

Case 3a: $|\delta(S) \cap \text{Path}(1', \ldots, k')| = k - 1$

If k is even, then, necessarily, $1' \in S$, contradicting the fact that

$$|\delta(S) \cap \text{Path}(1, 1', \ldots, k')| = 2s + 1.$$

If k is odd, then

$$S \cap \{1', 2', \ldots, k'\} = \{2', 4', \ldots, (k - 1)'\}$$

and

$$|\delta(S) \cap \text{Path}(1', 1, 2, \ldots, k)| = 2s + 1,$$

i.e., S is alternated along path $(1, 2, \ldots, k - 2)$; so there are f_k such roots.

Case 3b: $|\delta(S) \cap \text{Path}(1', \ldots, k')| = k - 2$

If k is even, then, necessarily, $1' \notin S$, contradicting the fact that

$$|\delta(S) \cap \text{Path}(1', 1, 2, \ldots, k)| = 2s + 2.$$

If k is odd, then, necessarily, $1'$, $(k - 1)' \in S$ and, since

$$|\delta(S) \cap \text{Path}(1', 1, \ldots, k)| = 2s + 2,$$

S is alternated along path $(2, 3, \ldots, k - 2)$, while S is pseudo-alternated along path $(1', \ldots, k')$; so there are $(k - 1)f_{k-1}$ such roots.

Case 4: Identical to Case 3, exchanging nodes i, i' for $i = 1, \ldots, k$.

Hence, the total number of roots is:

$$2k - 1 + f_{2k} + 2f_k + 2(k - 1)f_{k-1}$$
$$= 2k - 1 + f_{2k} + 2kf_{k-1} + 2f_{k-2} \quad \text{for } k \text{ odd}$$

and

$$2k - 1 + f_{2k} \qquad \text{for } k \text{ even,}$$

while the number of nonzero symmetric roots is f_k, stating Proposition 2(i).

We now prove Proposition 2(ii). It was proven in [8] that Par_{2k+1} is facet inducing for k odd and that it is not valid for k even. We now consider Par_{2k+1} for k even; the set of its roots is $R_1 \cup R_2$, where R_i denotes the set of roots of type i (Proposition 1), for $i = 1, 2$. To facilitate the computation of the rank of the set of roots, we use the following notion of *intersection vector*: for a subset S of $[1, k] \cup [1', k']$, define the vector $\pi(S)$ of $\{0, 1\}^{k(2k+1)}$ by

$$\pi(S)_{ij} = 1 \text{ if } i, j \in S \quad \text{and} \quad \pi(S)_{ij} = 0 \text{ otherwise}$$

for all i, j (not necessarily distinct) in $[1, k] \cup [1', k']$. Given a family of subsets $(S_a: a \in A)$ of $[1, k] \cup [1', k']$, the family of cut vectors $(\delta(S_a): a \in A)$ is linearly independent if and only if the family of intersection vectors $(\pi(S_a): a \in A)$ is linearly independent (see [8]).

First, we check that all roots in $R_1 = \{\delta(S_a): a \in A\}$ are linearly independent. For this, we take a linear combination of their intersection vectors:

$$\sum_{a \in A} \lambda_a \pi(S_a) = 0.$$

To verify that $\lambda_a = 0$ for all a, observe that, for each root $\delta(S_a)$ of R_1, one can find a pair (i, j) such that $\{i, j\} \subseteq S_a$, while $\{i, j\} \not\subseteq S_b$ for the other roots $\delta(S_b)$ of R_1 [for instance, take the pair $(k - 1, k)$ for the root $\delta(\{k, k - 1, k - 3, \ldots, 2, 1', 3', \ldots, k'\})$].

Next, we check that the rank of the family R_2 is

$$\binom{2k-1}{2} - 2k + 3.$$

For this, observe first that the subfamily R_2' of R_2 consisting of all possible singletons and pairs of $[1, k - 1] \cup [1', (k - 1)']$ has full rank equal to

$$2k - 2 + \binom{2k-2}{2} - (2k - 3) = \binom{2k-1}{2} - (2k - 3)$$

(easy if one considers the intersection vectors). Then, note that, for every cut $\delta(S)$ of R_2, nodes k, k' do not belong to S and S is alternated along Q, implying that

$$x_{kk} = x_{k'k'} = x_{kk'} = x_{ki} = x_{k'i} = x_{i,\, i+1} = 0$$

for $i \in [1, k - 1] \cup [1', (k - 1)']$, where $x = \pi(S)$ for $\delta(S) \in R_2$. Therefore, we deduce that the rank of R_2 is less than or equal to

$$\binom{2k+1}{2} - (6k - 4) = \binom{2k-1}{2} - (2k - 3).$$

Finally, we verify that the family $R_1 \cup R_2'$ is linearly independent, thus stating that the rank of face Par_{2k+1} for k even is

$$2k - 1 + \binom{2k-1}{2} - (2k - 3) = \binom{2k-1}{2} + 2.$$

Again, we take a linear combination of the intersection vectors

$$\sum \lambda_a \pi(S_a) + \sum \mu_c \pi(T_c) = 0,$$

where the first sum is over the intersection vectors corresponding to cuts in R_1 and the second one corresponds to cuts in R_2'. It is enough to show that $\lambda_a = 0$ for all a. For this, for the roots $\delta(S_a)$ of R_1 having $\{i, i + 1\} \subseteq S_a$ for some i, by looking at the coordinate $(i, i + 1)$ in the above linear combination we obviously obtain that $\lambda_a = 0$. For remaining roots $\delta(S_a)$ of R_1, looking at coordinate (k, i) with $i \in S_a$ also yields $\lambda_a = 0$. ∎

Given a vector $v = (v_{ij})_{1 \le i < j \le n}$ and two points, say 1 and n, the vector obtained from v by collapsing points 1 and n into the single point 1 is the vector $v' = (v'_{ij})_{1 \le i < j \le n-1}$ defined by

$$v'_{1i} = v_{1i} + v_{in} \text{ for } 2 \le i \le n - 1 \text{ and } v'_{ij} = v_{ij} \text{ for } 2 \le i < j \le n - 1.$$

The Fibonacci inequality Fib_{2k} can be obtained precisely by collapsing points k, k' into a single point $0'$ in the parachute inequality Par_{2k+1}. Using this observation, the roots of Fib_{2k} correspond to the roots of Par_{2k+1} of types 1 and 2. So, Fib_{2k} and Par_{2k+1}, for k even, have the same rank, but Fib_{2k} is valid while Par_{2k+1} is not. Observe also that Fib_{2k} coincides (up to renumerotation of the points) with the inequality obtained by collapsing in the bicycle odd wheel inequality (2) point 0 and one point of cycle C. From the above two facts follows the next result.

Proposition 3: The Fibonacci inequality Fib_{2k} is valid over the cut cone for any $k \geq 3$ and its rank is

$$\binom{2k-1}{2} + 2 = \binom{2k}{2} - 2k + 3.$$

Its roots are the cuts $\delta(S - \{k, k'\} + \{0'\})$ for S of type 1 and $\delta(S)$ for S of type 2. So, Fib_{2k} has $2k - 1 + f_{2k}$ roots and f_k nonzero symmetric roots.

3. Symmetries of the Parachute Inequality

The following two operations on facets of the cut cone C_n are given in [8]: (a) *permutation*—given a vector $v = (v_{ij})_{1 \leq i < j \leq n}$ and a permutation σ of $\mathrm{Sym}(n)$, set $v^{\sigma}_{ij} = v_{\sigma(i)\sigma(j)}$ for $1 \leq i < j \leq n$; then, inequality $v^{\sigma}.x \leq 0$ is said to be permutation equivalent to $v.x \leq 0$. (b) *switching*—given vector v and a root $\delta(S)$ of inequality $v.x \leq 0$, set $v^S_{ij} = -v_{ij}$ if $|S \cap \{i, j\}| = 1$ and $v^S_{ij} = v_{ij}$ otherwise; then, inequality $v^S.x \leq 0$ is said to be switching equivalent to $v.x \leq 0$. If inequality $v.x \leq 0$ is valid (resp. facet inducing) over the cut cone C_n, then both inequalities $v^{\sigma}.x \leq 0$, $v^S.x \leq 0$ are valid (resp. facet inducing) over C_n. In [7] it is shown that permutation and switching (by any cut) are the only symmetries of the cut polytope. The automorphism group $\mathrm{Aut}(v)$ of inequality $v.x \leq 0$ is the group $\{\sigma \in \mathrm{Sym}(n) : v^{\sigma} = v\}$ and its group $\mathrm{PS}(v)$ of double symmetries is the group $\{\sigma \in \mathrm{Sym}(n) : v^{\sigma} = v^S$ for some root $\delta(S)$ of $v.x \leq 0\}$; so $\mathrm{Aut}(v) \subseteq \mathrm{PS}(v)$ and $\mathrm{PS}(v)$ is the group of permutations which act simultaneously as switchings. So any facet yields many equivalent ones by switching and permutation. For instance, facet Par_7 yields precisely 7560 equivalent facets of C_7.

The example of facet Par_7 presents a lot of beautiful symmetries that we describe in more detail. The automorphism group of Par_7 is the subgroup of $\mathrm{Sym}(7)$ generated by the involution $\alpha = (11')(22')(33')$, so it is isomorphic to $\mathrm{Sym}(2)$. The group $\mathrm{PS}(\mathrm{Par}_7)$ of double symmetries of Par_7 is the dihedral group D_7.

Facet Par_7 has 21 roots (so it is a simplicial facet) partitioned into 3 classes:

$$R_a = \{\delta(a_i) : i \in [0, 6]\}, \; R_b = \{\delta(b_i) : i \in [0, 6]\},$$

and $\quad R_c = \{\delta(c_i) : i \in [0, 6]\}$,

where a_i for $i = 0, 1, \ldots, 6$ denote, respectively, the sets

$$\phi, \{2\}, \{2'\}, \{1, 3, 2'\}, \{1', 3', 2\}, \{2, 1'\}, \{2', 1\},$$

b_i for $i = 0, 1, \ldots, 6$ denote, respectively, the sets

$$\{2, 2'\}, \{1'\}, \{1\}, \{2, 3, 1', 3'\}, \{2', 3', 1, 3\}, \{2, 3'\}, \{2', 3\},$$

and c_i for $i = 0, 1, \ldots, 6$ denote, respectively, the sets

$$\{1, 3, 1', 3'\}, \{1', 3\}, \{1, 3'\}, \{1', 3', 3\}, \{1, 3, 3'\},$$
$$\{1, 2, 3'\}, \{1', 2', 3\}.$$

Each class R_a, R_b, R_c is the union of four orbits of Aut(Par_7) (one of size 1 for the symmetric root and three of size 2). Denote by $F_a = \text{Par}_7$, F_b, F_c the facets obtained by switching Par_7 by the symmetrical roots a_0, b_0, c_0, respectively. The facets F_a, F_b, F_c are not permutation equivalent; however, they have the same automorphism group: $\{id, \alpha\}$.

We consider the following involutions:

$$\pi_1 = (03)(13')(1'2'),\ \pi_2 = \alpha\pi_1\alpha,\ \pi_3 = (02)(1'3')(32'),$$

$$\pi_4 = \alpha\pi_3\alpha,\ \pi_5 = (01)(21')(2'3'),\ \pi_6 = \alpha\pi_5\alpha.$$

Then, it turns out that, for $i \in [1, 6]$, the facet obtained by switching of Par_7 by root $\delta(a_i)$ [resp. $\delta(b_i)$, $\delta(c_i)$] coincides with the facet obtained by permutation of Par_7 by π_i. Therefore, Par_7 has three nonpermutation equivalent switchings. Its group of double symmetries is the dihedral group D_7 with generators α, π_i for $1 \le i \le 6$.

Finally, we mention two more curiosities on the roots of Par_7:

(a) all subsets of $\{1, 1', 2, 2', 3, 3'\}$ can be generated by taking symmetric differences of members of the set $\{a_i: i \in [1, 6]\}$, or of $\{b_i: i \in [1, 6]\}$, or of $\{c_i: i \in [1, 6]\}$.

(b) c_0 is the complement of $b_0 \Delta \{0\}$, $c_i = b_i \Delta \{x\}$ with $x = 3, 3', 2, 2', 1, 1'$, for $i = 1, 2, 3, 4, 5, 6$, respectively.

Most of the above symmetries are lost for the parachute facet Par_{2k+1} with $k \ge 5$, k odd. The automorphism group of Par_{2k+1} is still the group of order 2 generated by the involution

$$\prod_{1 \le i \le k} (ii').$$

The number of orbits of the set of roots of Par_{2k+1} is:

$$3f_k/2 + f_{2k}/2 + (k-1)f_{k-1} + k$$

(number of symmetric roots plus one-half of number of nonsymmetric roots). It is known that the number of orbits of the set of roots is an upper bound for the number of nonpermutation equivalent switchings (see [7]); we conjecture that equality holds for Par_{2k+1}, k odd, $k \ge 5$ (but equality does not hold for Par_7).

4. Concluding Remarks

It turns out that both the parachute inequality and the bicycle odd wheel inequality can be decomposed as integer combination of triangle inequalities with all coefficients +1 except one coefficient -1. For instance, the parachute facet Par_{2k+1} for odd k can be decomposed as follows:

$$\text{Par}_{2k+1}.x = \sum_{1 \le i \le k-1} (T(a_{i'}, i, i+1) + T(a_i, i', (i+1)')) + T(0, 1, 1') - T(0, k, k'),$$

where $a_i = k$ for i odd and $a_i = a_{i'} = 0$ for i even, and

$$T(a, b, c) = x_{bc} - x_{ab} - x_{ac}$$

denotes the left-hand side of the triangle inequality on nodes a, b, c. A nice property of inequalities $v.x \le 0$ which can be "triangulated" is that $v.\delta(S)$ is even for all cuts $\delta(S)$. On the other hand, the Fibonacci face Fib_{2k} is the sum of triangles; for instance, for k even, we have:

$$\text{Fib}_{2k}.x = \sum_{1 \le i \le (k-2)/2} (T(0, 2i, 2i+1) + T(0, (2i)', (2i+1)') + T(0', (2i-1)', (2i)') + T(0, 1, 1').$$

Furthermore, we checked that any parachute, Fibonacci, or bicycle odd wheel inequality reduces, by consecutive collapsing, to some triangle inequality (the same holds for their switchings, see [6]).

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Author and Title Indexed for *The Fibonacci Quarterly*

Currently, Dr. Charles K. Cook of the University of South Carolina at Sumter is working on an AUTHOR index, TITLE index and PROBLEM index for *The Fibonacci Quarterly.* In fact, the three indices are already completed. We hope to publish these indices in 1993 which is the 30th anniversary of *The Fibonacci Quarterly.* Dr. Cook and I feel that it would be very helpful if the publication of the indices also had AMS classification numbers for all articles published in *The Fibonacci Quarterly.* We would deeply appreciate it if all authors of articles published in *The Fibonacci Quarterly* would take a few minutes of their time and send a list of articles with primary and secondary classification numbers to

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