ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

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Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

Dedication. This year's column is dedicated to **Dr. A. P. Hillman** in recognition of his 27 years of devoted service as editor of the Elementary Problems Section. Devotees of this column are invited to thank Abe by dedicating their next proposed problem to Dr. Hillman.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-706 *Proposed by K. T. Atanassov, Sofia, Bulgaria*

Prove that for $n \geq 0$,

$$\left(\frac{\pi e}{\pi + e}\right)^{1.4n} > F_n.$$

B-707 *Proposed by Herta T. Freitag, Roanoke, VA*

Consider a Pythagorean triple (a, b, c) such that

$$a = 2\sum_{i=1}^{n} F_i^2 \quad \text{and} \quad c = F_{2n+1}, \quad n \geq 2.$$

Prove or disprove that b is the product of two Fibonacci numbers.

B-708 *Proposed by Joseph J. Kostal, University of Illinois at Chicago, IL*

Find the sum of the series

$$\sum_{k=1}^{\infty} \frac{3^k F_k - 2^k L_k}{6^k}.$$

B-709 *Proposed by Alejando Necochea, Pan American University, Edinburg, TX*

Express $\displaystyle \frac{1}{n!}\frac{d^n}{dt^n}\left[\frac{t}{1 - t - t^2}\right]_{t=0}$ in terms of Fibonacci numbers.

B-710 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let P_n be the n^{th} Pell number, defined by $P_0 = 0$, $P_1 = 1$, $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$. Prove that

(a) $P_{3n+1} \equiv L_{3n+1} \pmod 5$,

(b) $P_{3n+2} \equiv -L_{3n+2} \pmod 5$.

(c) Find similar congruences relating Pell numbers and Fibonacci numbers.

B-711 *Proposed by Mihály Bencze, Sacele, Romania*

Let r be a natural number. Find a closed form expression for

$$\prod_{k=1}^{\infty}\left(1 - \frac{L_{4r}}{k^4} + \frac{1}{k^8}\right).$$

SOLUTIONS

Edited by A. P. Hillman

Fibonacci Analogues

B-680 *Proposed by Russell Jay Hendel & Sandra A. Monteferrante, Dowling College, Oakdale, NY*

For an integer $a \geq 0$, define a sequence, $x_0, x_1, \ldots$ by $x_0 = 0$, $x_1 = 1$, and $x_{n+2} = ax_{n+1} + x_n$ for $n \geq 0$. Let $d = (a^2 + 4)^{1/2}$. For $n \geq 2$, what is the nearest integer to dx_n?

Solution by H.-J. Seiffert, Berlin, Germany

If $a = 0$, then x_n is 0 for n even and is 1 for n odd. If $a = 1$, then (x_n) is the sequence of Fibonacci numbers. 2 is the nearest integer to $\sqrt{5}F_n = dx_n$ for $n = 2$ and $x_{n-1} + x_{n+1} = F_{n-1} + F_{n+1} = L_n$ is the nearest integer to $\sqrt{5}F_n = dx_n$ for $n \geq 3$ (see B-659). Now let $a \geq 2$ and $n \geq 2$. It is well known that $x_n = (b^n - c^n)/d$ and $x_{n-1} + x_{n+1} = b^n + c^n$, where $b = (a + d)/2$ and $c = (a - d)/2$. $a \geq 2$ implies the inequalities

(1) $\quad |c| < 1 \quad$ and $\quad -1/4 < c^2 = ac + 1 < 1/4.$

Thus, we have

$$\begin{aligned}|dx_n - (x_{n-1} + x_{n+1})| &= |b^n - c^n - (b^n + c^n)|\\ &= 2|c|^n \leq 2|c|^2 = 2|ac + 1| < 1/2.\end{aligned}$$

This shows that $x_{n-1} + x_{n+1}$ is the nearest integer to dx_n.

Also solved by R. André-Jeannin, Paul S. Bruckman, Guo-Gang Gao, Lawrence Somer, and the proposers.

Straight Line Separating (F_n, F_{n+1}) from (F_{n+1}, F_{n+2})

B-684 *Proposed by L. Kuipers, Sierre, Switzerland*

(a) Find a straight line in the Cartesian plane such that (F_n, F_{n+1}) and (F_{n+1}, F_{n+2}) are on opposite sides of the line for all positive integers n.

(b) Is the line unique?

Solution by Paul S. Bruckman, Edmonds, WA

Let $P_n = (F_n, F_{n+1})$, $n = 1, 2, \ldots$, denote points in the Cartesian plane. We use the identity

(1) $\quad F_{n+1} = \alpha F_n + \beta^n, \; n = 1, 2, \ldots .$

Since β^n and β^{n+1} have opposite signs (and are necessarily nonzero), we see that $(F_{n+1} - \alpha F_n)$ and $(F_{n+2} - \alpha F_{n+1})$ have opposite signs. This implies that P_n and P_{n+1} lie on opposite sides of the line L, defined by

(2) $\quad L = \{(x, y) : y = \alpha x\}.$

Therefore, L satisfies the conditions of part (a).

Suppose $L' = \{(x, y) : y = rx + s\}$ is any line with this same property, where r and s are real. Since L' must intersect the segment P_nP_{n+1} for each n, the following inequalities must hold:

(3) $\quad rF_n + s > F_{n+1}, \quad rF_{n+1} + s < F_{n+2}, \quad n = 1, 3, 5, \ldots .$

Then, using (1), $(\alpha - r)F_n + \beta^n < s < (\alpha - r)F_{n+1} + \beta^{n+1}$, or

(4) $\quad \dfrac{s - \beta^{n+1}}{F_{n+1}} < \alpha - r < \dfrac{s - \beta^n}{F_n}, \quad n = 1, 3, 5, \ldots .$

Taking limits in (4) as $n \to \infty$, we see that either end of the inequalities approaches 0; therefore, $r = \alpha$. Moreover, we must have

(5) $\quad \beta^n < s < \beta^{n+1}, \quad n = 1, 3, 5, \ldots .$

Again taking limits as $n \to \infty$ in (5), we conclude that $s = 0$. Thus, we conclude that $L' = L$, i.e., the desired line L is uniquely described by (2).

Also solved by Charles Ashbacher, Piero Filipponi, C. Georghiou, Russell Jay Hendel, Hans Kappus, Y. H. Harris Kwong, Mohammad Parvez Shaikh, Lawrence Somer, and the proposer.

Approximation to k as a Function of F_k

B-685 *Proposed by Stanley Rabinowitz, Westford, MA, and Gareth Griffith, University of Saskatchewan, Saskatoon, Saskatchewan, Canada*

For integers $n \geq 2$, find k as a function of n such that

$$F_{k-1} \leq n < F_k.$$

Solution by Lawrence Somer, Washington, D.C.

It follows from the Binet formula for F_k that

$$\sqrt{5}F_k + \beta^k = \alpha^k.$$

Note that $0 < \beta^k < .5$ if $k \geq 2$ is even and $-.5 < \beta^k < 0$ if $k \geq 3$ is odd. Thus, it follows that if $k \geq 2$ is even, then F_k is the largest integer m such that

(1) $\quad \alpha^{k-1} < \sqrt{5}m < \alpha^k.$

It also follows that if $k \geq 3$ is odd, then F_k is the smallest integer m such that

(2) $\quad \alpha^k < \sqrt{5}m < \alpha^{k+1}.$

Using (1) and (2) and taking logarithms to the base α, we have that

$$k = [\log_\alpha(\sqrt{5}n + .5)] + 1,$$

where $[x]$ denotes the greatest integer less than or equal to x.

Also solved by Paul S. Bruckman, Piero Filipponi, C. Georghiou, Russell Jay Hendel, and the proposers.

Some Nearly Geometric Progressions

B-686 *Proposed by Jeffrey Shallit, U. of Waterloo, Ontario, Canada*

Let a and b be integers with $0 < a \le b$. Set $c_0 = a$, $c_1 = b$, and for $n \ge 2$ define c_n to be the least integer with $c_n/c_{n-1} > c_{n-1}/c_{n-2}$. Find a closed form for c_n in the cases:

(a) $a = 1$, $b = 2$; (b) $a = 2$, $b = 3$.

Solution by C. Georghiou, University of Patras, Patras, Greece

(a) We will show that $c_n = F_{2n+1}$. Suppose that it is true for $n = k - 1$, and $n = k$. Then, from the definition of c_n we have

$$c_{k+1} = [c_k^2/c_{k-1}] + 1$$

where, as usual, $[x]$ denotes the greatest integer less than or equal to x.

Using the known identity $F_{2k+3}F_{2k-1} - F_{2k+1}^2 = 1$, we get

$$c_k^2/c_{k-1} = F_{2k+1}^2/F_{2k-1} = F_{2k+3} - 1/F_{2k-1},$$

from which it follows that $c_{k+1} = F_{2k+3}$. The proof is completed by noting that the assertion is true for $n = 0$ and $n = 1$.

(b) In a similar way we show that $c_n = 2^n + 1$. Assuming that

$$c_{k-1} = 2^{k-1} + 1 \quad \text{and} \quad c_k = 2^k + 1,$$

we get

$$c_{k+1} = [c_k^2/c_{k-1}] + 1 = [(2^{2k} + 2^{k+1} + 1)/(2^{k-1} + 1)] + 1 = 2^{k+1} + 1$$

and at the same time

$$c_0 = 2^0 + 1 \quad \text{and} \quad c_1 = 2^1 + 1.$$

Also solved by Paul S. Bruckman, Russell Euler, Herta Freitag, Russell Jay Hendel, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

B-687 *Proposed by Jeffrey Shallit, University of Waterloo, Ontario, Canada*

Let c_n be as in Problem B-686. Find a closed form for c_n in the case with $a = 1$ and b an integer greater than 1.

Solution by Lawrence Somer, Washington, D.C.

Let $\{H_n\}$ denote the second-order linear recurrence which has initial terms $H_0 = 1$, $H_1 = b$ and satisfies the recursion relation

$$H_{n+2} = (b + 1)H_{n+1} - (b - 1)H_n.$$

We claim that $c_n = H_n$. Clearly, $c_0 = H_0$ and $c_1 = H_1$. To prove our result, it suffices to show that

(1) $\quad H_{n+2}/H_{n+1} > H_{n+1}/H_n$

and

(2) $\quad (H_{n+2} - 1)/H_{n+1} \le H_{n+1}/H_n.$

Thus, it suffices to prove that

(3) $\quad H_{n+2}H_n > H_{n+1}^2$

and

(4) $\quad H_{n+2}H_n \le H_{n+1}^2 + H_n.$

One can easily show by induction that

(5) $\quad H_n \ge b^n$

and

(6) $\quad H_{n+2}H_n - H_{n+1}^2 = (b - 1)^n.$

Thus, from (6), we have that

(7) $\quad H_{n+2}H_n = H_{n+1}^2 + (b - 1)^n > H_{n+1}^2,$

which establishes inequality (3). From (5) and (7), we obtain

$$H_{n+2}H_n = H_{n+1}^2 + (b - 1)^n \le H_{n+1}^2 + b^n \le H_{n+1}^2 + H_n,$$

which establishes inequality (4). Hence $c_n = H_n$. The closed form for H_n is obtained using standard recursion theory.

Also solved by Paul S. Bruckman, C. Georghiou, Russell Jay Hendel, and the proposer.

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