ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

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Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

Dedication. This year's column is dedicated to Dr. A. P. Hillman in recognition of his 27 years of devoted service as editor of the Elementary Problems Section. Devotees of this column are invited to thank Abe by dedicating their next proposed problem to Dr. Hillman.

BASIC FORMULAS

The Fibonacci numbers \( F_n \) and the Lucas numbers \( L_n \) satisfy

\[
F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1;
\]

\[
L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.
\]

Also, \( \alpha = (1 + \sqrt{5})/2, \beta = (1 - \sqrt{5})/2, F_n = (\alpha^n - \beta^n)/\sqrt{5}, \) and \( L_n = \alpha^n + \beta^n. \)

PROBLEMS PROPOSED IN THIS ISSUE

B-706 Proposed by K. T. Atanassov, Sofia, Bulgaria

Prove that for \( n \geq 0, \)

\[
\left( \frac{n^4 + e}{n + e} \right)^{1.4n} > F_n.
\]

B-707 Proposed by Herta T. Freitag, Roanoke, VA

Consider a Pythagorean triple \((a, b, c)\) such that

\[
a = 2 \sum_{i=1}^{n} F_i^2 \quad \text{and} \quad c = F_{2n+1}, \quad n \geq 2.
\]

Prove or disprove that \( b \) is the product of two Fibonacci numbers.

B-708 Proposed by Joseph J. Kostal, University of Illinois at Chicago, IL

Find the sum of the series

\[
\sum_{k=1}^{ } \frac{3^kF_k^2 - 2^kL_k}{6^k}.
\]

B-709 Proposed by Alejandro Necochea, Pan American University, Edinburg, TX

Express \( \frac{1}{n!} \frac{d^n}{dt^n} \left[ \frac{t}{1 - t - t^2} \right]_{t=0} \) in terms of Fibonacci numbers.

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B-710 Proposed by H.-J. Seiffert, Berlin, Germany

Let \( P_n \) be the \( n \)th Pell number, defined by \( P_0 = 0, P_1 = 1, P_{n+2} = 2P_{n+1} + P_n \) for \( n \geq 0 \). Prove that

(a) \( P_{3n+1} \equiv L_{3n+1} \pmod{5} \),

(b) \( P_{3n+2} \equiv -L_{3n+2} \pmod{5} \).

(c) Find similar congruences relating Pell numbers and Fibonacci numbers.

B-711 Proposed by Mihály Bencze, Sacele, Romania

Let \( r \) be a natural number. Find a closed form expression for

\[
\prod_{k=1}^{n} \left( 1 - \frac{L_{4r}}{k^4} + \frac{1}{k^8} \right)
\]

SOLUTIONS

Edited by A. P. Hillman

Fibonacci Analogue

B-680 Proposed by Russell Jay Hendel & Sandra A. Monteferrante, Dowling College, Oakdale, NY

For an integer \( a \geq 0 \), define a sequence, \( x_0, x_1, \ldots \) by \( x_0 = 0, x_1 = 1, \) and \( x_{n+2} = ax_{n+1} + x_n \) for \( n \geq 0 \). Let \( d = (a^2 + 4)^{1/2} \). For \( n \geq 2 \), what is the nearest integer to \( dx_n \)?

Solution by H.-J. Seiffert, Berlin, Germany

If \( a = 0 \), then \( x_n \) is 0 for \( n \) even and is 1 for \( n \) odd. If \( a = 1 \), then \( (x_n) \) is the sequence of Fibonacci numbers. 2 is the nearest integer to \( \sqrt{5}x_n = dx_n \) for \( n = 2 \) and \( x_{n-1} + x_{n+1} = F_{n-1} + F_{n+1} = L_n \) is the nearest integer to \( \sqrt{5}x_n = dx_n \) for \( n \geq 3 \) (see B-659). Now let \( a \geq 2 \) and \( n \geq 2 \). It is well known that \( x_n = (b^n - \sigma^n)/d \) and \( x_{n-1} + x_{n+1} = b^n + \sigma^n \), where \( b = (a + d)/2 \) and \( \sigma = (a - d)/2 \). \( a \geq 2 \) implies the inequalities

\[
\text{1) } |\sigma| < 1 \text{ and } -1/4 < \sigma^2 = a\sigma + 1 < 1/4.
\]

Thus, we have

\[
|dx_n - (x_{n-1} + x_{n+1})| = |b^n - \sigma^n - (b^n + \sigma^n)| = 2|\sigma|^n \leq 2|\sigma|^2 = 2|a\sigma + 1| < 1/2.
\]

This shows that \( x_{n-1} + x_{n+1} \) is the nearest integer to \( dx_n \).

Also solved by R. André-Jeannin, Paul S. Bruckman, Guo-Gang Gao, Lawrence Somer, and the proposers.

Straight Line Separating \( (F_n, F_{n+1}) \) from \( (F_{n+1}, F_{n+2}) \)

B-684 Proposed by L. Kuipers, Sierre, Switzerland

(a) Find a straight line in the Cartesian plane such that \( (F_n, F_{n+1}) \) and \( (F_{n+1}, F_{n+2}) \) are on opposite sides of the line for all positive integers \( n \).

(b) Is the line unique?
Solution by Paul S. Bruckman, Edmonds, WA

Let \( P_n = (F_n, F_{n+1}) \), \( n = 1, 2, \ldots \), denote points in the Cartesian plane. We use the identity

\[
F_{n+1} = aF_n + \beta^n, \quad n = 1, 2, \ldots
\]

Since \( \beta^n \) and \( \beta^{n+1} \) have opposite signs (and are necessarily nonzero), we see that \( (F_{n+1} - aF_n) \) and \( (F_{n+2} - aF_{n+1}) \) have opposite signs. This implies that \( F_n \) and \( F_{n+1} \) lie on opposite sides of the line \( L \), defined by

\[
L = \{ (x, y) : y = ax \}.
\]

Therefore, \( L \) satisfies the conditions of part (a).

Suppose \( L' = \{ (x, y) : y = rx + s \} \) is any line with this same property, where \( r \) and \( s \) are real. Since \( L' \) must intersect the segment \( F_nF_{n+1} \) for each \( n \), the following inequalities must hold:

\[
rF_n + s > F_{n+1}, \quad rF_{n+1} + s < F_{n+2}, \quad n = 1, 3, 5, \ldots
\]

Then, using (1), \( (a-r)F_n + \beta^n < s < (a-r)F_{n+1} + \beta^{n+1} \), or

\[
\frac{\beta^n}{F_{n+1}} < \frac{s - \beta^n}{F_n}, \quad n = 1, 3, 5, \ldots
\]

Taking limits in (4) as \( n \to \infty \), we see that either end of the inequalities approaches 0; therefore, we must have

\[
\beta^n < s < \beta^{n+1}, \quad n = 1, 3, 5, \ldots
\]

Again taking limits as \( n \to \infty \) in (5), we conclude that \( s = 0 \). Thus, we conclude that \( L' = L \), i.e., the desired line \( L \) is uniquely described by (2).

Also solved by Charles Ashbacher, Piero Filipponi, C. Georgiou, Russell Jay Hendel, Hans Kappus, Y. H. Harris Kwong, Mohammad Parvez Shaikh, Lawrence Somer, and the proposer.

Approximation to \( k \) as a Function of \( F_k \)

B-685 Proposed by Stanley Rabinowitz, Westford, MA, and Gareth Griffith, University of Saskatchewan, Saskatoon, Saskatchewan, Canada

For integers \( n \geq 2 \), find \( k \) as a function of \( n \) such that

\[
F_{k-1} \leq n < F_k.
\]

Solution by Lawrence Somer, Washington, D.C.

It follows from the Binet formula for \( F_k \) that

\[
\sqrt{5}F_k + \beta^k = a^k.
\]

Note that \( 0 < \beta^k < .5 \) if \( k \geq 2 \) is even and \( -.5 < \beta^k < 0 \) if \( k \geq 3 \) is odd. Thus, it follows that if \( k \geq 2 \) is even, then \( F_k \) is the largest integer \( m \) such that

\[
a^{k-1} < \sqrt{5}m < a^k.
\]

It also follows that if \( k \geq 3 \) is odd, then \( F_k \) is the smallest integer \( m \) such that

\[
a^k < \sqrt{5}m < a^{k+1}.
\]

Using (1) and (2) and taking logarithms to the base \( a \), we have that
\[ k = \lceil \log_a (\sqrt{5}n + .5) \rceil + 1, \]
where \( \lceil x \rceil \) denotes the greatest integer less than or equal to \( x \).

Also solved by Paul S. Bruckman, Piero Filipponi, C. Georghiou, Russell Jay Hendel, and the proposers.

**Some Nearly Geometric Progressions**

**B-686** Proposed by Jeffrey Shallit, U. of Waterloo, Ontario, Canada

Let \( a \) and \( b \) be integers with \( 0 < a \leq b \). Set \( c_0 = a, c_1 = b, \) and for \( n \geq 2 \) define \( c_n \) to be the least integer with \( c_n/c_{n-1} > c_{n-1}/c_{n-2} \). Find a closed form for \( c_n \) in the cases:

(a) \( a = 1, b = 2; \) (b) \( a = 2, b = 3. \)

Solution by C. Georghiou, University of Patras, Patras, Greece

(a) We will show that \( c_n = F_{2k+1} \). Suppose that it is true for \( n = k - 1, \) and \( n = k \). Then, from the definition of \( c_n \) we have

\[ c_{k+1} = \lceil \frac{c_k^2}{c_{k-1}} \rceil + 1 \]

where, as usual, \( \lceil x \rceil \) denotes the greatest integer less than or equal to \( x. \)

Using the known identity \( F_{2k+3}F_{2k-1} - F_{2k+1}^2 = 1, \) we get

\[ \frac{c_k^2}{c_{k-1}} = F_{2k+2}^2, \]

from which it follows that \( c_{k+1} = F_{2k+3}. \) The proof is completed by noting that the assertion is true for \( n = 0 \) and \( n = 1. \)

(b) In a similar way we show that \( c_n = 2^n + 1. \) Assuming that \( c_{k-1} = 2^{k-1} + 1 \) and \( c_k = 2^k + 1, \)
we get

\[ c_{k+1} = \lceil \frac{c_k^2}{c_{k-1}} \rceil + 1 = \left\lceil \frac{(2^{2k} + 2^k + 1)(2^{k-1} + 1)}{2^{k-1} + 1} \right\rceil + 1 = 2^{k+1} + 1 \]

and at the same time

\[ c_0 = 2^0 + 1 \] and \( c_1 = 2^1 + 1. \)

Also solved by Paul S. Bruckman, Russell Euler, Herta Freitag, Russell Jay Hendel, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

**B-687** Proposed by Jeffrey Shallit, University of Waterloo, Ontario, Canada

Let \( c_n \) be as in Problem B-686. Find a closed form for \( c_n \) in the case with \( a = 1 \) and \( b \) an integer greater than 1.

Solution by Lawrence Somer, Washington, D.C.

Let \( \{H_n\} \) denote the second-order linear recurrence which has initial terms \( H_0 = 1, H_1 = b \) and satisfies the recursion relation

\[ H_{n+2} = (b + 1)H_{n+1} - (b - 1)H_n. \]

We claim that \( c_n = H_n \). Clearly, \( c_0 = H_0 \) and \( c_1 = H_1. \) To prove our result, it suffices to show that

\[ [Feb.
(1) \( \frac{H_{n+2}}{H_{n+1}} > \frac{H_{n+1}}{H_n} \)

and

(2) \( \frac{(H_{n+2} - 1)}{H_{n+1}} \leq \frac{H_{n+1}}{H_n} \).

Thus, it suffices to prove that

(3) \( H_{n+2}H_n > H_{n+1}^2 \)

and

(4) \( H_{n+2}H_n \leq H_{n+1}^2 + H_n \).

One can easily show by induction that

(5) \( H_n \geq b^n \)

and

(6) \( H_{n+2}H_n - H_{n+1}^2 = (b - 1)^n \).

Thus, from (6), we have that

(7) \( H_{n+2}H_n = H_{n+1}^2 + (b - 1)^n > H_{n+1}^2 \),

which establishes inequality (3). From (5) and (7), we obtain

\[ H_{n+2}H_n = H_{n+1}^2 + (b - 1)^n \leq H_{n+1}^2 + b^n \leq H_{n+1}^2 + H_n, \]

which establishes inequality (4). Hence \( a_n = H_n \). The closed form for \( H_n \) is obtained using standard recursion theory.

Also solved by Paul S. Bruckman, C. Georghiou, Russell Jay Hendel, and the proposer.

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