FIBONACCI NUMBERS AND LADDER NETWORK IMPEDANCE

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0. Introduction

This work can be considered the natural extension of a previous study about the same subject. In fact, the authors have studied [4], from a mathematical point of view, a particular numerical triangle, called DFF, characterizing the transfer function of an electrical ladder network formed by a cascade of N identical coupled cells.

The present paper deals with the study of another new triangle named DFFz from the authors' initials and from the fact that it characterizes the equivalent impedances determination of the same type of electrical network. In particular, this triangle is strictly related to Thevenin's equivalent impedance which can be expressed by the ratio of two polynomials: the one related to DFFz and the other to DFF triangle.

The DFFz triangle is shown to have a noteworthy interest from the mathematical point of view, because some of its properties are connected with Fibonacci numbers.

1. The Generating Polynomials

The DFFz triangle can be formed in the following manner (with $a_{n,k}$ being the general coefficient).

We define [3]:

(1.1) $a_{n,k} = 0 \quad \text{if } n < k$

(1.2) $a_{n,k} = 1 \quad \text{if } n = k$

(1.3) $a_{n,k} = n + 1 \quad \text{if } k = 0$

while the other elements of the triangle can be derived from the recursive formula

(1.4) $$a_{n,k} = a_{n-1,k} + \sum_{\alpha=0}^{n-1} a_{\alpha,k-1} \quad \text{if } n > k.$$

In this manner, we have the DFFz triangle for values of $a_{n,k}$:

$n \backslash k$	0	1	2	3	4	5	6 ...
0	1						
1	2	1					
2	3	4	1				
3	4	10	6	1			
4	5	20	21	8	1		
5	6	35	56	36	10	1	
6	7	56	126	120	55	12	1
...	...	...	...	...	...	...	...

Thus, for example, $a_{2,1} = 4$ and $a_{6,2} = 126$.

The generating polynomial $P_n(x)$ is defined [1] as

$$(1.5)\qquad P_n(x) = \sum_{k=0}^{n} a_{n,k}x^k,$$

where

$$(1.6)\qquad a_{n,k} = \left.\frac{D^k P_n(x)}{k!}\right|_{x=0}$$

From the DFFz triangle, it is possible to obtain the expressions of the polynomial for small values of n, namely,

$$(1.7)\qquad \begin{aligned} P_0(x) &= 1 \\ P_1(x) &= 2 + x \\ P_2(x) &= 3 + 4x + x^2 \\ P_3(x) &= 4 + 10x + 6x^2 + x^3 \end{aligned}$$

and so on.

From (1.1), (1.2), (1.3), (1.4), and (1.5), we have

$$(1.8)\qquad \sum_{k=1}^{n} a_{n,k}x^k = \sum_{k=1}^{n} a_{n-1,k}x^k + \sum_{k=1}^{n}\sum_{\alpha=0}^{n-1} a_{\alpha,k-1}x^k.$$

From (1.8), using (1.1), (1.2), (1.3), and (1.5), we have

$$(1.9)\qquad P_n(x) - a_{n,0} = P_{n-1}(x) - a_{n-1,0} + x\sum_{\alpha=0}^{n-1}\sum_{k=1}^{\alpha+1} a_{\alpha,k-1}x^{k-1}$$

$$(1.10)\qquad P_n(x) - (n+1) = P_{n-1}(x) - n + x\sum_{\alpha=0}^{n-1} P_\alpha(x)$$

$$(1.11)\qquad P_n(x) = 1 + P_{n-1}(x) + x\sum_{\alpha=0}^{n-1} P_\alpha(x)$$

which is the recursive formula for the polynomials.

With the initial condition $P_0(x) = 1$, it is easy to obtain the polynomials (1.7). Furthermore, we can also use (1.6) to find the triangle coefficients.

In order to find the polynomials, we must apply the previous method. Let

$$(1.12)\qquad f(x, t) = \sum_{n=1}^{\infty} P_n(x)t^n.$$

Then

$$(1.13)\qquad P_n(x) = \left.\frac{D^n[f(x, t)]}{n!}\right|_{t=0}$$

From (1.11) and (1.12), we have

$$\begin{aligned}(1.14)\qquad f(x, t) &= \sum_{n=1}^{\infty}[1 + P_{n-1}(x)]t^n + x\sum_{n=1}^{\infty}\sum_{\alpha=0}^{n-1} P_\alpha(x)t^n \\ &= t\sum_{n=1}^{\infty}[1 + P_{n-1}(x)]t^{n-1} + x\sum_{n=1}^{\infty} t^n[P_0(x) + P_1(x) + \cdots + P_{n-1}(x)] \\ &= t\sum_{n=1}^{\infty} P_{n-1}(x)t^{n-1} + \sum_{n=1}^{\infty} t^n + x[1 + f(x, t)]\frac{t}{1-t} \\ &= t[1 + f(x, t)] + \frac{t}{1-t} + x[1 + f(x, t)]\frac{t}{1-t} \\ &= \frac{-t^2 + t(2+x)}{t^2 - t(2+x) + 1}.\end{aligned}$$

If we develop the denominator in (1.14) in partial fractions, we obtain

$$(1.15)\quad f(x,\ t) = \frac{1/y(x)}{t - b(x)/2} + \frac{-1/y(x)}{t - c(x)/2} - 1,$$

where

$$y(x) = (x^2 + 4x)^{1/2},\ b(x) = 2 + x + y, \text{ and } c(x) = 2 + x - y.$$

From the binomial expansion in (1.15), and after simplification, we also have

$$(1.16)\quad f(x,\ t) = \frac{-2}{yb(x)}\left\{1 + \sum_{n \ge 1}\left[\frac{t}{b(x)/2}\right]^n\right\} + \frac{2}{yc(x)}\left\{1 + \sum_{n \ge 1}\left[\frac{t}{c(x)/2}\right]^n\right\} - 1$$

$$= \sum_{n \ge 1}\left\{\frac{2}{yc(x)}\left[\frac{t}{c(x)/2}\right]^n - \frac{2}{yb(x)}\left[\frac{t}{b(x)/2}\right]^n\right\}$$

$$= \sum_{n \ge 1}\frac{2^{n+1}}{y(x)}\left[\frac{1}{[c(x)]^{n+1}} - \frac{1}{[b(x)]^{n+1}}\right]t^n,$$

from which we have, using (1.12):

$$(1.17)\quad P_n(x) = \frac{2^{n+1}}{\sqrt{x^2 + 4x}}\left\{\frac{1}{[2 + x - \sqrt{x^2 + 4x}]^{n+1}} - \frac{1}{[2 + x + \sqrt{x^2 + 4x}]^{n+1}}\right\}.$$

Furthermore, considering the binomial expansion, we are able to put $P_n(x)$ in the following better way:

$$(1.18)\quad P_n(x) = \frac{1}{2^{n+1}}\left\{\left(\frac{2 + x}{\sqrt{x^2 + 4x}} + 1\right)\sum_{h=0}^{n}\binom{n}{h}(x + 2)^{n-h}y^h - \left(\frac{2 + x}{\sqrt{x^2 + 4x}} - 1\right)\sum_{h=0}^{n}(-1)^h\binom{n}{h}(x + 2)^{n-h}y^h\right\}.$$

From this equation, on distinguishing the case of odd h from that of even h, we can write

$$(1.19)\quad P_n(x) = \frac{1}{2^n}\left[\sum_{h \equiv 0 \pmod 2}^{n}\binom{n}{h}(x + 2)^{n-h}x^{h/2}(x + 4)^{h/2} + \sum_{h \equiv 1 \pmod 2}^{n}\binom{n}{h}(x + 2)^{n-h+1}x^{(h-1)/2}(x + 4)^{(h-1)/2}\right].$$

2. Determination of $a_{n,k}$

From (1.6), we have

$$(2.1)\quad a_{n,k} = \frac{1}{k!2^n}\left[\sum_{h \equiv 0 \pmod 2}^{n}\binom{n}{h}\sum_{j=0}^{k}\binom{k}{j}D^j[x^{h/2}(x + 4)^{h/2}] \cdot D^{k-j}[x + 2]^{n-h} + \sum_{h \equiv 1 \pmod 2}^{n}\binom{n}{h}\sum_{j=0}^{k}\binom{k}{j}D^j[x^{(h-1)/2}(x + 4)^{(h-1)/2}] \cdot D^{k-j}[x + 2]^{n-h+1}\right]_{x=0}.$$

Considering Leibniz's formula, we may write

$$(2.2)\quad D^j[x^{h/2}(x + 4)^{h/2}] = \sum_{m=0}^{j}\binom{j}{m}\binom{h/2}{m}m!x^{(h/2)-m} \cdot \binom{h/2}{j - m}(j - m)!(x + 4)^{(h/2)-j+m}.$$

$$(2.3)\qquad D^j[x^{(h-1)/2}(x+4)^{(h-1)/2}] = \sum_{m=0}^{j}\binom{j}{m}\binom{(h-1)/2}{m}m!\,x^{((h-1)/2)-m} \cdot \binom{(h-1)/2}{j-m}(j-m)!\,(x+4)^{((h-1)/2)-j+m}.$$

$$(2.4)\qquad D^{k-j}[(x+2)^{n-h}] = \binom{n-h}{k-j}(k-j)!\,(x+2)^{n-h-k+j}.$$

$$(2.5)\qquad D^{k-j}[(x+2)^{n-h+1}] = \binom{n-h+1}{k-j}(k-j)!\,(x+2)^{n-h+1-k+j}.$$

From equations (2.3), (2.4), and (2.5), and from the properties of binomial coefficients, (2.2) becomes

$$(2.6)\qquad a_{n,k} = \frac{1}{2^n}\left\{\sum_{h\equiv 0 \pmod 2}^{n}\binom{n}{h}\sum_{j=0}^{k}\sum_{m=0}^{j}x^{(h/2)-m}\binom{n-h}{k-j}(x+2)^{n-h-k+j} \cdot \binom{h/2}{m}\binom{h/2}{j-m}(x+4)^{(h/2)-j+m} + \sum_{h\equiv 1 \pmod 2}^{n}\binom{n}{h}\sum_{j=0}^{k}\sum_{m=0}^{j}\binom{(h-1)/2}{m}\binom{(h-1)/2}{j-m}(x+4)^{((h-1)/2)-j+m} \cdot x^{((h-1)/2)-m}\binom{n-h+1}{k-j}(x+2)^{n-h+1-k+j}\right\}_{x=0}.$$

When $x = 0$, the m-sums (which contain the x-term) exist if and only if $m = (h-1)/2$ and $m = h/2$, respectively. So we can write

$$(2.7)\qquad a_{n,k} = \sum_{h\equiv 0 \pmod 2}^{n}\binom{n}{h}\sum_{j=0}^{k}\binom{h/2}{j-h/2}\binom{n-h}{k-j}2^{h-k-j} + \sum_{h\equiv 1 \pmod 2}^{n}\binom{n}{h}\sum_{j=0}^{k}\binom{(h-1)/2}{j-(h-1)/2}\binom{n-h+1}{k-j}2^{h-k-j-1}.$$

Equation (2.7) is the wanted expression which permits us to determine $a_{n,k}$ by substituting for n and k.

3. The Properties of $a_{n,k}$

3.1 The row sums of the triangle are equal to Fibonacci numbers with even subscripts

From the expression of $P_n(x)$, when $x = 1$, we have

$$(3.1)\qquad P_n(1) = \frac{1}{2^{n+1}\sqrt{5}}[(3+\sqrt{5})^{n+1} - (3-\sqrt{5})^{n+1}].$$

From Binet's formula, we have

$$(3.2)\qquad F_{2n+2} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n+2}\right].$$

If we notice that

$$(3.3)\qquad \left(\frac{1\pm\sqrt{5}}{2}\right)^2 = \frac{3\pm\sqrt{5}}{2},$$

then (3.2) becomes

$$(3.4)\qquad F_{2n+2} = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{3-\sqrt{5}}{2}\right)^{n+1}\right].$$

In this manner, we have

$$P_n(1) = F_{2n+2},$$

where $F_0 = 0$, $F_2 = 1$, $F_4 = 3$, ...

3.2 The sums of the triangle diagonals give the powers of 2

From a direct inspection of DFFz triangle and (1.4), we have that the sum of the elements of an upward-slanting diagonal is equal to the sum of all elements which are above this diagonal PLUS ONE and, consequently, it is equal to the sum of all superior upward-slanting diagonals plus one. This sum value is a power of 2.

In fact, if we define

$$\sum^n = \sum_{r=0}^{n} a_{n-r,\,r},$$

it is possible to write

$$\begin{aligned}\sum^n &= \sum^{n-1} + \sum^{n-2} + \cdots + \sum^1 + 1 + 1\\ &= 2\left(\sum^{n-2} + \sum^{n-3} + \cdots + \sum^1 + 2\right)\\ &= \cdots\\ &= 2^{n-2}\left(\sum^1 + 2\right)\\ &= 2^n,\end{aligned}$$

since $\sum^1 = 2$.

4. Conclusions

The principal aim of this paper has been the determination of a closed expression of the general coefficient $a_{n,k}$ of a new numerical triangle, named DFFz, which characterizes Thevenin's equivalent impedance of a ladder network whose elementary cells are directly coupled. Moreover, the authors present some of the interesting mathematical properties of the triangle, one of which is connected with Fibonacci numbers.

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