# ON SOME NUMBER SEQUENCES RELATED TO THE PARITY OF BINOMIAL COEFFICIENTS

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## 1. Pascal's Triangle Mod 2

It is well known that striking patterns of triangles can be produced from Pascal's triangle by replacing each binomial coefficient by its residue with respect to a certain modulus. The arrays thus produced were considered by various authors; see, for instance, Gould [5], Gardner [1], Long [10], or Sved [17]. For example, *Pascal's triangle mod 2* (Fig. 1) is the array of zeros and ones obtained by considering the parity of each entry in the usual Pascal triangle. It can be readily constructed using the basic recursion formula

$$(1.1)\qquad \binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

together with the rules for addition mod 2. (In Fig. 1, this array is shown "right-justified" for convenience in further discussions, with all entries resulting from coefficients of the form $\binom{n}{n}$ aligned in the rightmost column. Furthermore, for the sake of clarity, groups of zeros in this figure have been enclosed within triangular shapes.)

We shall be concerned in this paper with some number sequences introduced via Pascal's triangle mod 2. Gould [5] has considered the sequence obtained by reading the rows of this array as base two representations of numbers. We shall introduce analogously other number sequences and show how certain regularities of such sequences follow directly from the patterns found within the array. It is our purpose in this paper to base our discussion essentially on the geometrical structure of Pascal's triangle mod 2. So we complete this introduction with a description of this geometrical structure.

We borrow the following terminology from Sved [16, 17], with the notation $\left|{n \atop r}\right|$ representing the residue of $\binom{n}{r}$ mod 2 ($0 \le r \le n$). The *cluster of order h*, or *h-cluster*, is the portion of the array formed by all the residues $\left|{n \atop r}\right|$ for $0 \le n < 2^h$, and the *zero-hole of order h* ($h \ne 0$), or *h-hole*, is the (inverted, left-justified) triangular array made of $(2^h - 1)$ decreasing rows of zeros, with $(2^h - 1)$ entries in the first row down to a single entry in its last row. [Anticipating the forthcoming geometrical characterization of Pascal's triangle mod 2, see the following paragraph, the $h$-hole thus corresponds to all residues $\left|{n \atop r}\right|$ with

$$(1.2)\qquad 2^h \le n < 2^{h+1} - 1 \quad\text{and}\quad n - (2^h - 1) \le r < 2^h.]$$

For example, the clusters of orders 0, 1, and 2 are, respectively,

```
                      1
                    1 1
        1         1 0 1
   1,  1 1,  and 1 1 1 1,
```

while the zero-holes of orders 1 and 2 are of the form

```
   0   and   0 0 0.
             0 0
             0
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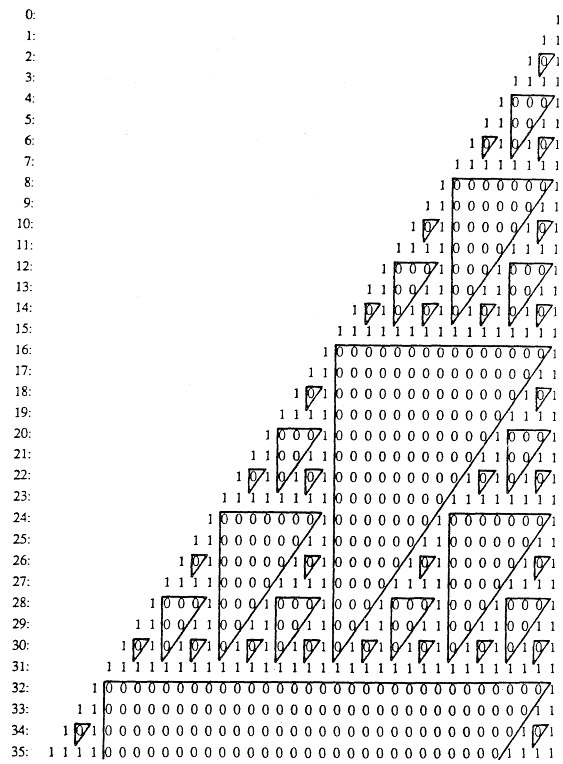


**Figure 1**

The overall structure of Pascal's triangle mod 2 can be described as follows. Let us observe the array as it grows downward, thus producing successive (nested) clusters. *Then the cluster of order $h$, consisting of rows 0 down to $2^h - 1$, is made of three clusters of order $(h - 1)$ surrounding a zero-hole of order $(h - 1)$* (see Fig. 2, where the three $(h - 1)$-clusters have been labeled, respectively, I, II, and III). A formal proof of this characterization can be given by induction, using the recursion formula (1.1) (see Sved [16]). The geometrical pattern of the array could become even more striking by replacing all zeros by blanks in Figure 1. Note that when extending the process to an infinite number of rows, the limiting pattern is found to be "self-similar" with fractal dimension $\log_2 3$, as discussed in Wolfram [19] (see also the "Sierpinski gasket" described in Mandelbrot [12, p. 142]).

This geometrical characterization of Pascal's triangle mod 2 allows us to state a few basic properties.

(1.i) Row $2^h - 1$ consists of $2^h$ ones: 111...111 ($2^h$ 1's).

(1.ii) Row $2^h$ consists of two ones separated by $2^h - 1$ zeros: 100...001 ($2^h - 1$ 0's).

(1.iii) More generally, row $2^h + u$, $0 \le u < 2^h$, consists of two copies of row $u$ separated by $(2^h - 1 - u)$ zeros.

Result (1.i) follows from the fact that row $2^h - 1$, which is the bottom row of the $h$-cluster, is obtained by concatenating the bottom row of the $(h - 1)$-cluster with itself. Property (1.ii) is then an easy consequence of (1.i) using mod 2 addition. As for property (1.iii), it expresses the fact that row $2^h + u$, which is located in the $(h + 1)$-cluster, is obtained by inserting, in between two copies of row $u$ of the $h$-cluster, the $(u+1)^{\text{th}}$ row of the $h$-hole [i.e., a sequence of $(2^h - 1 - u)$ zeros]. These results can be rephrased as follows in terms of the parity of binomial coefficients.

(1.i') All coefficients $\binom{2^h - 1}{r}$, $0 \le r \le 2^h - 1$, are odd.

(1.ii') Coefficients $\binom{2^h}{r}$ are odd only for $r = 0$ and $2^h$.

(1.iii') $\left|\begin{matrix} 2^h + u \\ r \end{matrix}\right| = \left|\begin{matrix} 2^h + u \\ 2^h + r \end{matrix}\right| = \left|\begin{matrix} u \\ r \end{matrix}\right|$ for $0 \le u < 2^h$ and $0 \le r \le u$.

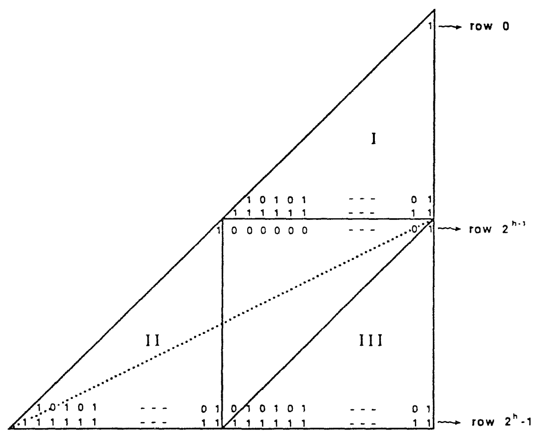


**Figure 2. The cluster of order $h$**
(The dotted line indicates the "principal diagonal" $\Delta_{2^h - 1}$.)

As was observed by Kung [9] or Sved [17], results (1.i)-(1.iii') follow from a simple glance at the binomial array mod 2. However, the reader should note that all six of these properties can also be obtained as immediate consequences of certain well-known facts about binomial coefficients. For instance, by a result due to Kummer [8, p. 116], one has the following:

(1.iv) The exponent of 2 in the prime factorization of $\binom{n}{r}$ equals the number of borrows in the subtraction $n - r$ in base two.

(See Singmaster [14] or Goetgheluck [4] for recent proofs.) Hence $\left|\begin{smallmatrix} n \\ r \end{smallmatrix}\right| = 1$ if and only if there are no borrows in this subtraction. A direct algebraic proof of property (1.i) can be found in Vinogradov [18, p. 20]. Alternately, as observed by Roberts [13], (1.i) follows immediately from the fact that, for a fixed $n$, the number of odd binomial coefficients $\binom{n}{r}$ is given by $2^{\#_1(n)}$, where $\#_1(n)$ represents the number of 1's appearing in the base two representation of $n$. This last result, stated in Glaisher [3], is easily justified using the following theorem of Lucas [11]:

$$\binom{n}{r} \equiv \binom{n_k}{r_k}\binom{n_{k-1}}{r_{k-1}} \cdots \binom{n_0}{r_0} \pmod 2,$$

where $(n_k n_{k-1} \ldots n_0)_{\text{two}}$ and $(r_k r_{k-1} \ldots r_0)_{\text{two}}$ are the binary representations of $n$ and $r$, respectively. (This last result of Lucas plays a central role in the "masking" relation used by Jones & Matijasevic [7] for encoding the history of calculations of a Turing machine. It is this latter work which has prompted the present author's interest in Pascal's triangle mod 2.)

## 2. Gould's Numbers

Let us now use the binomial array just discussed to define a sequence $\{G_n\}_{n \geq 0}$ of natural numbers as follows: $G_n$ is the number whose binary representation is given by the $n^{\text{th}}$ row of Pascal's triangle mod 2. This sequence, which starts

$$1, 3, 5, 15, 17, 51, 85, 255, 257, 771, 1285, \ldots,$$

was considered by Gould [5] (see Sloane [15], sequence no. 988). We shall call the $G_n$'s *Gould's numbers*.

We can use facts (1.i)-(1.iii) about Pascal's triangle mod 2 to deduce some basic relationships among Gould's numbers. For instance, we have

$$(2.1) \qquad G_{2^h} = 2^{2^h} + 1 = F_h,$$

where $F_h$ denotes the $h^{\text{th}}$ Fermat number. This stems immediately from the particular form of row $2^h$ [see (1.ii) above]. [Similarly, by (1.i), $G_{2^h - 1} = 2^{2^h} - 1$.] It then readily follows that for an arbitrary $n = 2^h + u$, $0 \leq u < 2^h$, we have

$$(2.2) \qquad G_{2^h + u} = G_{2^h} \cdot G_u,$$

since the sequence of 1's and 0's forming row $2^h + u$, as described in (1.iii), can be directly seen as being the (binary) product of row $2^h$ and row $u$.

For $n$ having the binary representation $(n_k n_{k-1} \ldots n_0)_{\text{two}}$, one then deduces from (2.1) and (2.2) the remarkable relation

$$(2.3) \qquad G_n = \prod_{i=0}^{k} F_i^{n_i}.$$

Indeed, writing $n$ as a sum of powers of two, the digits $n_i$ indicate the powers $2^i$ needed for expressing $n$. Result (2.3) was stated by Gould [5] [see formula (50)] and a proof was given by Hewgill [6]. (Gould [5] stated another remarkable relation about Gould's numbers, namely: $G_{2n+1} = 3G_{2n}$. This result is easily proved inductively from the geometrical pattern of the binomial array mod 2. A formal proof of the same result can be found in Garfinkel & Selkow [2].)

When (2.2) is rewritten in the form

$$(2.4) \qquad G_{2^h + u} = G_u 2^{2^h} + G_u,$$

one obtains nice symmetrical representations. For instance, (2.4) yields the following for $h = 3$ and $0 \leq u < 8$:

$$\begin{aligned}
G_8 &= 257 = 1 \cdot 256 + 1\\
G_9 &= 771 = 3 \cdot 256 + 3\\
G_{10} &= 1{,}285 = 5 \cdot 256 + 5\\
G_{11} &= 3{,}855 = 15 \cdot 256 + 15\\
G_{12} &= 4{,}369 = 17 \cdot 256 + 17\\
G_{13} &= 13{,}107 = 51 \cdot 256 + 51\\
G_{14} &= 21{,}845 = 85 \cdot 256 + 85\\
G_{15} &= 65{,}535 = 255 \cdot 256 + 255
\end{aligned}$$

A suggestive interpretation of Gould's numbers can also be obtained from (2.4), using property (1.iii'). Let us recall that by definition the $G_n$'s satisfy the equality

$$(2.5) \qquad G_n = \sum_{r=0}^{n} \left|\begin{matrix} n \\ r \end{matrix}\right| 2^{n-r}.$$

Then (2.4) says that for $n = 2^h + u$ this sum can be seen as made of two parts corresponding, respectively, to the successive values $0, 1, \ldots, u$ and $2^h, 2^h + 1, \ldots, 2^h + u = n$ of the index $r$. In the former case one has, because of (1.iii'),

$$\sum_{r=0}^{u} \left|\begin{matrix} n \\ r \end{matrix}\right| 2^{n-r} = \sum_{r=0}^{u} \left|\begin{matrix} u \\ r \end{matrix}\right| 2^{n-r} = \left( \sum_{r=0}^{u} \left|\begin{matrix} u \\ r \end{matrix}\right| 2^{u-r} \right) \cdot 2^{2^h},$$

so this partial sum corresponds to $G_u$ shifted by a factor of $2^{2^h}$. In the latter case, the terms sum directly to $G_u$ because, for $r = 2^h + i$, $0 \le i \le u$, one can write, again by (1.iii'),

$$\left|\begin{matrix} n \\ r \end{matrix}\right| 2^{n-r} = \left|\begin{matrix} u \\ i \end{matrix}\right| 2^{u-i}.$$

Figure 1 nicely illustrates the situation, since it has been displayed in such a way that in each $h$-cluster, the column corresponding to the term of weight $2^{2^h}$ in the binary expansion of $G_{2^h+u}$ is easily located.

## 3. Along Fibonacci Diagonals

We now want to use the binomial array mod 2 to introduce other sequences of numbers. Among remarkable lines in the (standard) Pascal triangle are the *Fibonacci diagonals*, i.e., those slant lines whose entries sum to consecutive terms of the Fibonacci sequence. When the binomial coefficients are displayed in the shape of a right-justified triangle, similar to Figure 1, the $n^{\text{th}}$ Fibonacci diagonal, starting at $\binom{n}{0}$, contains all those entries obtained by moving successively two columns to the right and one row up. By the basic formula (1.1), the $n^{\text{th}}$ Fibonacci number $f_n$ (where $f_0 = f_1 = 1$ and $f_{n+2} = f_{n+1} + f_n$) is the sum of all coefficients thus obtained:

$$f_n = \sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n-r}{r},$$

with $[x]$ indicating the integer part of $x$. For instance,

$$f_9 = \binom{9}{0} + \binom{8}{1} + \binom{7}{2} + \binom{6}{3} + \binom{5}{4} = 1 + 8 + 21 + 20 + 5 = 55.$$

In analogy with the way Gould's numbers were defined, we now want to introduce a sequence $\{H_n\}_{n \ge 0}$ of natural numbers whose binary representations are given by the Fibonacci diagonals in Pascal's triangle mod 2. Let us use $\Delta_n$, $n \ge 0$, to represent the string of digits found along the $n^{\text{th}}$ Fibonacci diagonal mod 2 (for instance, $\Delta_9 = 10101$). Then $H_n$ is the number represented in base two by $\Delta_n$. We thus have, analogously to (2.5),

$$H_n = \sum_{r=0}^{\left[\frac{n}{2}\right]} \left|\begin{matrix} n-r \\ r \end{matrix}\right| 2^{\left[\frac{n}{2}\right]-r} \tag{3.1}$$

so that, e.g., $H_9 = (10101)_{\text{two}} = 21$. The first values of the $H_n$-sequence are:

1, 1, 3, 2, 7, 5, 13, 8, 29, 21, 55, 34, 115, 81, 209, 128, 465, 337, 883, ... .

As might be expected, the $H_n$'s satisfy some nice generation rules, and we will make use of the geometry of Pascal's triangle mod 2 to give proofs of these rules. Before doing so, however, it is interesting to redisplay the entries of Figure 1 so that the $n^{\text{th}}$ diagonal $\Delta_n$ becomes the $n^{\text{th}}$ row in the new array (see Fig. 3). Some striking patterns can be observed in this array.

For instance, it is readily seen, for those values listed, that $\Delta_{2^h-1}$ consists of a one followed by a string of $2^{h-1} - 1$ zeros, so that

$$H_{2^h-1} = 2^{2^{h-1}-1}.$$

Also, the staircase pattern of Figure 3 appears to be made of symmetrical parts that could be directly described by introducing a terminology based on "clusters" and "holes," as was done for Figure 1. For example, rows 0 down to 14 can be seen as being separated in two layers by row 7; the lower layer (rows 8 to 14) is made of three sections, namely a copy of the upper "cluster" (rows 0 to 6) that has been shifted down by 8 rows and to the left by 4 columns, next to an upside down copy of this same part (i.e., a mirror image of rows 0-6 in row 7), the remaining entries in between these blocks being filled with a "hole" of zeros. Thus, $\Delta_9$ consists of a copy of $\Delta_1$ (1) and a copy of $\Delta_5$ (101) separated by one zero, while $\Delta_{13}$ consists, conversely, of $\Delta_5$ and $\Delta_1$ separated by three zeros.

```
 0:                                 1
 1:                                 1
 2:                               1 1
 3:                               1 0
 4:                             1 1 1
 5:                             1 0 1
 6:                           1 1 0 1
 7:                           1 0 0 0
 8:                         1 1 1 0 1
 9:                         1 0 1 0 1
10:                       1 1 0 1 1 1
11:                       1 0 0 0 1 0
12:                     1 1 1 0 0 1 1
13:                     1 0 1 0 0 0 1
14:                   1 1 0 1 0 0 0 1
15:                   1 0 0 0 0 0 0 0
16:                 1 1 1 0 1 0 0 0 1
17:                 1 0 1 0 1 0 0 0 1
18:               1 1 0 1 1 1 0 0 1 1
19:               1 0 0 0 1 0 0 0 1 0
20:             1 1 1 0 0 1 1 0 1 1 1
21:             1 0 1 0 0 0 1 0 1 0 1
22:           1 1 0 1 0 0 0 1 1 1 0 1
23:           1 0 0 0 0 0 0 0 1 0 0 0
24:         1 1 1 0 1 0 0 0 0 1 1 0 1
25:         1 0 1 0 1 0 0 0 0 0 1 0 1
26:       1 1 0 1 1 1 0 0 0 0 0 1 1 1
27:       1 0 0 0 1 0 0 0 0 0 0 0 1 0
28:     1 1 1 0 0 1 1 0 0 0 0 0 0 1 1
29:     1 0 1 0 0 0 1 0 0 0 0 0 0 0 1
30:   1 1 0 1 0 0 0 1 0 0 0 0 0 0 0 1
31:   1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
32: 1 1 1 0 1 0 0 0 1 0 0 0 0 0 0 0 1
33: 1 0 1 0 1 0 0 0 1 0 0 0 0 0 0 0 1
```

Figure 3

Such observations about the geometrical pattern of Figure 3 will be made more precise in the following sections, where the main results of this paper will be established.

## 4. The Principal Diagonals

Going back to Pascal's triangle mod 2 (Fig. 1), we call the Fibonacci diagonal $\Delta_{2^h-1}$ the *principal diagonal* of the cluster of order $h$. It thus consists of all entries of the form

$$\left|\begin{matrix} 2^h - 1 - r \\ r \end{matrix}\right| \text{ for } 0 \le r \le 2^{h-1} - 1.$$

We now prove a few basic properties of principal diagonals.

*Lemma 4.1:* $\Delta_{2^h-1} = 100\ldots0$ $(2^{h-1} - 1$ zeros).

This result could be obtained directly from property (1.iv): it suffices to note that

$$\left|\begin{matrix} 2^h - 1 - r \\ r \end{matrix}\right| = 0, \text{ unless } r = 0,$$

since, for $r > 0$, a borrow certainly occurs when subtracting $r$ from $2^h - 1 - r$ (in base two representations). The following is an alternate proof, based solely on the geometrical observations introduced above. We use the notation $\Delta_{n,r}$ for the $r^{\text{th}}$ element $\left|{n-r \atop r}\right|$ of string $\Delta_n$, $0 \le r \le \left[\frac{n}{2}\right]$, so that

$$\Delta_n = \Delta_{n,0}\Delta_{n,1} \ldots \Delta_{n,\left[\frac{n}{2}\right]}.$$

*Proof:* The elements $\Delta_{2^h-1,r}$, $0 \le r \le 2^{h-1} - 1$, can be separated in two groups, according to the value of the index $r$.

(i) $0 \le r \le 2^{h-2} - 1$. By property (1.iii'), we have

$$\Delta_{2^h-1,r} = \left|{2^h - 1 - r \atop r}\right| = \left|{2^{h-1} + (2^{h-1} - 1 - r) \atop r}\right| = \left|{2^{h-1} - 1 - r \atop r}\right|$$
$$= \Delta_{2^{h-1}-1,r}$$

so that the portion of $\Delta_{2^h-1}$ corresponding to the given range of $r$ is identical to the principal diagonal of the $(h - 1)$-cluster.

(ii) $2^{h-2} \le r \le 2^{h-1} - 1$. It is readily checked that the bounding conditions (1.2), as modified for the zero-hole of order $(h - 1)$, are satisfied by the two components of each entry $\Delta_{2^h-1,r}$. Thus, this portion of $\Delta_{2^h-1}$ is entirely included in the $(h - 1)$-hole.

The proof of the Lemma can then be completed by an easy induction on $h$. The base case can be read directly from Figure 3 and the induction step follows from (i) and (ii): the first portion of $\Delta_{2^h-1}$, which by the induction hypothesis is of the form $100\ldots0$ ($2^{h-2} - 1$ zeros), gets juxtaposed to the second portion made of $2^{h-2}$ zeros, so to give the desired form for the principal diagonal of the $h$-cluster. ∎

It follows from the preceding proof that the principal diagonal $\Delta_{2^h-1}$ of the cluster of order $h$ goes through this cluster in a very regular way. For instance, when $r = 2^{h-2}$, we get the entry

$$\left|{2^{h-1} + 2^{h-2} - 1 \atop 2^{h-2}}\right|:$$

this tells us that the diagonal $\Delta_{2^h-1}$ "enters" the $(h - 1)$-hole at the first entry of its middle row. Similarly, for $r = 2^{h-1} - 1$, we have the entry

$$\left|{2^{h-1} \atop 2^{h-1} - 1}\right|,$$

so that the last element of $\Delta_{2^h-1}$ is found at the end of the first row of the $(h - 1)$-hole. When combined with the fact that the principal diagonal starts at the leftmost entry of the cluster, this information on specific entries of $\Delta_{2^h-1}$ leads to the dotted line of Figure 2, which represents this principal diagonal. We now want to make explicit certain types of symmetry within the $h$-cluster connected to the principal diagonal.

It is trivially true that each line of the Pascal triangle mod 2 is symmetrical (with respect to its middle), i.e., remains the same when inverted from left to right: this indeed is even true of the (standard) Pascal triangle itself, because of the basic relationship

$$\binom{n}{r} = \binom{n}{n-r}.$$

We now want to prove a symmetry property concerning the columns. We show that the principal diagonal $\Delta_{2^h-1}$ can be seen as an "axis of vertical symmetry" for the portions of the columns determined by the $h$-cluster, in the sense that the entries above and under the principal diagonal, on each such portion of column, are pairwise identical. For instance, the principal diagonal $\Delta_{15}$ of the 4-cluster cuts it in such a way that the section of column 5 included in this cluster is separated into 110-011 by $\Delta_{15}$, and that column 12 is separated into 100010-0-010001 (with the middle 0 belonging to $\Delta_{15}$).

*Lemma 4.2:* The cluster of order $h$ is "vertically symmetrical" with respect to its principal diagonal $\Delta_{2^h-1}$.

*Proof:* The proof is by induction on $h$, with basic cases being easily verified. Let us consider, for a given $r$ such that $0 \le r \le 2^h - 1$, the portion of the $r^{\text{th}}$ column inside the $h$-cluster (which we shall call abusively the "$r^{\text{th}}$ column of the $h$-cluster"). There are then two possible cases:

(i) $0 \le r \le 2^{h-1}-1$. By the geometry of the binomial array mod 2, column $r$, which is entirely included in the $(h - 1)$-cluster II (see Fig. 2), is a copy of the analogous column of the $(h - 1)$-cluster I, so the symmetry property follows from case (i) of the proof of Lemma 4.1 and from the induction hypothesis.

(ii) $2^{h-1} \le r \le 2^h - 1$. Column $r$ then consists of three parts: a vertical string in the $(h - 1)$-cluster I and another one in III separated by a vertical section of the $(h - 1)$-hole (see Fig. 2). Clearly, again because of the geometry of the array, the parts in I and III are identical and each is self-symmetrical, by the induction hypothesis. It thus remains to show that $\Delta_{2^h-1}$ cuts the $(h - 1)$-hole symmetrically. But this is an easy consequence of the fact just mentioned above that $\Delta_{2^h-1}$ enters the $(h - 1)$-hole at the first element of the middle row of this hole and ends at the last element of the first row. ■

A consequence of Lemma 4.2 is that the symmetry with respect to the principal diagonal $\Delta_{2^h-1}$ inside the $h$-cluster can also be seen as acting along diagonal lines, in the sense that two strings "parallel" and "equidistant" to $\Delta_{2^h-1}$ will be identical. For instance, $\Delta_{12} = 1110011$, whose first entry is $\left|\begin{smallmatrix}12\\0\end{smallmatrix}\right|$, is identical to the string determined by the line of the same slope starting at $\left|\begin{smallmatrix}15\\3\end{smallmatrix}\right|$. More generally, any entry $\left|\begin{smallmatrix}v\\0\end{smallmatrix}\right|$, with $v < 2^h - 1$, which is located at the top of column $2^h - 1 - v$ of the $h$-cluster, is surely equal to the bottom entry on this same column inside the $h$-cluster, namely,

$$\left|\begin{matrix}2^h - 1\\2^h - 1 - v\end{matrix}\right|.$$

Now, if we issue from these two entries two lines parallel to the diagonal $\Delta_{2^h-1}$, we will obtain identical strings, because we then encounter pairs of entries, located on same columns, which are equidistant from the principal diagonal, hence equal by Lemma 4.2. We thus have

*Lemma 4.3:* The cluster of order $h$ is "obliquely symmetrical" with respect to its principal diagonal $\Delta_{2^h-1}$. ■

## 5. The Geometry of $\Delta_n$

Given $n = 2^h + u$ with $0 \le u < 2^h - 1$, we present in this section some rules for expressing the $n^{\text{th}}$ Fibonacci diagonal $\Delta_n$ in terms of diagonals depending on $u$ (note that for $u = 2^h - 1$, the rule for $\Delta_n$ is given by Lemma 4.1 above). This diagonal $\Delta_{2^h+u}$, which contains the entries

$$\Delta_{2^h+u,\,r} = \left|\begin{matrix}2^h + u - r\\ r\end{matrix}\right|, \text{ for } 0 \le r \le \left[\frac{2^h + u}{2}\right] = 2^{h-1} + \left[\frac{u}{2}\right],$$

can be found in the cluster of order $(h + 1)$, starting at entry $\left|\begin{smallmatrix}2^h+u\\0\end{smallmatrix}\right|$ and moving upward diagonally. It thus consists of three parts corresponding, respectively, to the regions of the $(h + 1)$-cluster being cut by this diagonal (see

Fig. 4): the *head* of $\Delta_{2^h+u}$ is made of all the entries $\Delta_{2^h+u,r}$ belonging to the $h$-cluster II, the *body* is the part included in the $h$-hole and the *tail* comes from the $h$-cluster I. We now prove some basic results about the Fibonacci diagonal $\Delta_{2^h+u}$.

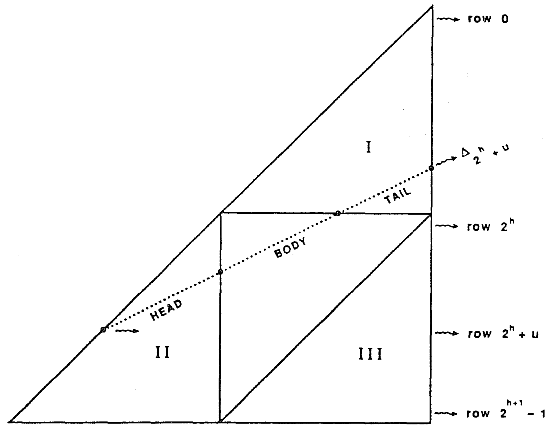


**Figure 4. The Fibonacci diagonal $\Delta_{2^h+u}$ in the $(h + 1)$-cluster**

*Lemma 5.1:* The head, the body, and the tail of $\Delta_{2^h+u}$ consist, respectively, of the entries $\Delta_{2^h+u,r}$ such that

a) head: $0 \le r \le \left[\frac{u}{2}\right]$

b) body: $\left[\frac{u}{2}\right] + 1 \le r \le u$

c) tail: $u + 1 \le r \le 2^{h-1} + \left[\frac{u}{2}\right]$.

*Proof:* The verification involves routine calculations. For example, the range of $r$ for the head follows from the stipulation to stay inside II and from the slope of Fibonacci diagonals being 1/2. For other cases, we need to identify the values of $r$ for which conditions (1.2) are satisfied. For instance, for $r = [u/2] + 1$, we get

$$\Delta_{2^h+u,\left[\frac{u}{2}\right]+1} = \left| \begin{matrix} 2^h + u - \left[\frac{u}{2}\right] - 1 \\ \left[\frac{u}{2}\right] + 1 \end{matrix} \right|$$

and (1.2) can easily be verified for $0 < u < 2^h - 1$. The value of $r$ can be increased up to $u$ while remaining in the $h$-hole, and we then get

$$\Delta_{2^h+u,u} = \left| \begin{matrix} 2^h \\ u \end{matrix} \right|$$

which again satisfies (1.2). But, for $r = u + 1$, we have

$$\Delta_{2^h+u,u+1} = \left| \begin{matrix} 2h - 1 \\ u + 1 \end{matrix} \right|$$

which is above the zero-hole of order $h$ and thus in the tail of $\Delta_{2^h+u}$. To complete the proof, we just note that in the case $u = 0$, the body is void since the head then consists of the single element $\Delta_{2^h,0} = 1$, which is the first entry of row $2^h$, located at the apex of the $h$-cluster II, while the next element $\Delta_{2^h,1} = 1$ is in I and thus belongs to the tail. ∎

*Lemma 5.2:* The head and the tail of $\Delta_{2^h+u}$ are, respectively, $\Delta_u$ and $\Delta_{2^h-2-u}$.

*Proof:* For $r$ such that $0 \le r \le \left[\frac{u}{2}\right]$, we have

$$\Delta_{2^h+u,r} = \left|\begin{matrix} 2^h + u - r \\ r \end{matrix}\right| = \left|\begin{matrix} u - r \\ r \end{matrix}\right| = \Delta_{u,r},$$

where the second equality is true by (1.iii'), since we have $0 \le u - r \le 2^h$ and $0 \le r \le u - r$. This shows that the head of $\Delta_{2^h+u}$ is $\Delta_u$. In the case of the tail, let us notice that its first entry, namely,

$$\Delta_{2^h+u,u+1} = \left|\begin{matrix} 2^h - 1 \\ u + 1 \end{matrix}\right|,$$

is located at the bottom of column $u + 1$ inside the $h$-cluster I. From the discussion preceding Lemma 4.3, the top entry of this column is

$$\left|\begin{matrix} 2^h - 2 - u \\ 0 \end{matrix}\right|,$$

i.e., the first element of $\Delta_{2^h-2-u}$. We then conclude by the "oblique symmetry" of Lemma 4.3. ∎

It should be noted here that our assumption that $u < 2^h - 1$ ensures that $2^h - 2 - u \ge 0$.

Before closing this section, we would like to comment further on the relationship between the entries making up the tail of $\Delta_{2^h+u}$ and the diagonal $\Delta_{2^h-2-u}$. By Lemma 5.1(c), these entries are of the general form

$$\Delta_{2^h+u,r} = \left|\begin{matrix} 2^h + u - r \\ r \end{matrix}\right|, \text{ for } r = u + 1, \ldots, \left[\frac{2^h + u}{2}\right]. \tag{5.1}$$

We just noted in the proof of Lemma 5.2 that for the first of these entries we can write, by "oblique symmetry,"

$$\Delta_{2^h+u,u+1} = \left|\begin{matrix} 2^h - 1 \\ u + 1 \end{matrix}\right| = \left|\begin{matrix} 2^h - 2 - u \\ 0 \end{matrix}\right|.$$

More generally, for $r$ ranging over the values indicated in (5.1), this same "oblique symmetry" described in Lemma 4.3 gives us

$$\left|\begin{matrix} 2^h + u - r \\ r \end{matrix}\right| = \left|\begin{matrix} 2^h - 2 - u - t \\ t \end{matrix}\right|, \tag{5.2}$$

where $t = r - u - 1$, i.e., $t$ takes on, successively, the values 0, 1, ... up to

$$\left[\frac{2^h + u}{2}\right] - u - 1.$$

Another equivalent way of expressing this relationship is that the element

$$\left|\begin{matrix} 2^h + u - r \\ r \end{matrix}\right|, \text{ with } u + 1 \le r \le \left[\frac{2^h + u}{2}\right] \tag{5.3}$$

can be directly rewritten, by a simple change of variable, as

$$\left|\begin{matrix} 2^h - 1 - s \\ u + 1 + s \end{matrix}\right|, \text{ where } 0 \le s \le \left[\frac{2^h + u}{2}\right] - u - 1 \tag{5.4}$$

(note that this last expression still represents an element of the diagonal $\Delta_{2^h+u}$). In turn, by the symmetry property of Lemma 4.3, entry (5.4) becomes an element of $\Delta_{2^h-2-u}$, namely,

$$\left|\begin{matrix} 2^h - 2 - u - t \\ t \end{matrix}\right|, \text{ where, again, } 0 \le t \le \left[\frac{2^h + u}{2}\right] - u - 1. \tag{5.5}$$

These remarks will be used in the proof of the next result.

## 6. Calculating the $H_n$'s

We are now in position to give some calculation rules for the numbers $H_n$.

*Proposition 6.1:* (i) $H_{2^h-1} = 2^{2^{h-1}-1}$.

(ii) $H_{2^h+u} = H_u \cdot 2^{2^{h-1}} + H_{2^h-2-u}$, for $0 \le u < 2^h - 1$.

*Proof:* Case (i) follows immediately from Lemma 4.1. The idea behind the proof of case (ii) is that the value of $H_{2^h+u}$ can be obtained in three steps by looking consecutively at the head, body, and tail of $\Delta_{2^h+u}$ as described in Lemmas 5.1-5.2. Taking into consideration the shifting of $\Delta_u$ when it becomes the head of $\Delta_{2^h+u}$, the result is then transparent.

To be more precise, let us evaluate, for $n = 2^h + u$, the three partial sums, $S_1$, $S_2$, and $S_3$, obtained from (3.1) according to the ranges of $r$ identified in Lemma 5.1. We first get:

$$S_1 = \sum_{r=0}^{\left[\frac{u}{2}\right]} \left|\begin{matrix} n - r \\ r \end{matrix}\right| 2^{\left[\frac{n}{2}\right]-r} = \sum_{r=0}^{\left[\frac{u}{2}\right]} \left|\begin{matrix} u - r \\ r \end{matrix}\right| 2^{2^{h-1}+\left[\frac{u}{2}\right]-r}$$

$$= \left(\sum_{r=0}^{\left[\frac{u}{2}\right]} \left|\begin{matrix} u - r \\ r \end{matrix}\right| 2^{\left[\frac{u}{2}\right]-r}\right) \cdot 2^{2^{h-1}} = H_u \cdot 2^{2^{h-1}},$$

where the second equality follows from (1.iii'). This first partial sum thus corresponds to the shifting of $\Delta_u$ by $2^{h-1}$ positions in order to get the head of $\Delta_{2^h+u}$.

The second partial sum is

$$S_2 = \sum_{r=\left[\frac{u}{2}\right]+1}^{u} \left|\begin{matrix} n - r \\ r \end{matrix}\right| 2^{\left[\frac{n}{2}\right]-r}.$$

It was already observed in the proof of Lemma 5.1 that, for the given values of $r$, we have

$$\left|\begin{matrix} 2^h + u - r \\ r \end{matrix}\right| = 0, \text{ so that } S_2 = 0.$$

Finally, we turn to the last partial sum, $S_3$. From the discussion surrounding expression (5.2) [or equivalently (5.3)-(5.5)], we can write

$$S_3 = \sum_{r=u+1}^{\left[\frac{n}{2}\right]} \left|\begin{matrix} n - r \\ r \end{matrix}\right| 2^{\left[\frac{n}{2}\right]-r} = \sum_{t=0}^{\left[\frac{n}{2}\right]-u-1} \left|\begin{matrix} 2^h - 2 - u - t \\ t \end{matrix}\right| 2^{\left[\frac{n}{2}\right]-u-1-t} = H_{2^h-2-u}.$$

The value of this partial sum thus corresponds to the tail of $\Delta_{2^h+u}$ being given, by "oblique symmetry," by $\Delta_{2^h-2-u}$. ■

Proposition 6.1, case (ii), provides us with nice symmetrical representations for the $H_n$'s, as was the case with Gould's numbers. For instance, for $h = 3$ and $0 \le u < 7$, we have the following expressions:

$$\begin{aligned} H_8 &= 29 = 1 \cdot 16 + 13 \qquad & H_{12} &= 115 = 7 \cdot 16 + 3 \\ H_9 &= 21 = 1 \cdot 16 + 5 & H_{13} &= 81 = 5 \cdot 16 + 1 \\ H_{10} &= 55 = 3 \cdot 16 + 7 & H_{14} &= 209 = 13 \cdot 16 + 1 \\ H_{11} &= 34 = 2 \cdot 16 + 2 & & \end{aligned}$$

By restricting the sequence $\{H_n\}_{n \geq 0}$, respectively, to elements of even and of odd ranks, we obtain the two subsequences

$$1,\ 3,\ 7,\ 13,\ 29,\ 55,\ 115,\ 209,\ 465,\ 883,\ \ldots$$

and

$$1,\ 2,\ 5,\ 8,\ 21,\ 34,\ 81,\ 128,\ 337,\ 546,\ \ldots\ .$$

To these sequences would correspond triangular arrays that could be obtained from Figure 3 by deleting appropriate alternate rows. The behavior of these new sequences is very close to that of the $H_n$'s and it is possible to deduce for them results entirely analogous to those presented in Lemmas 5.1-5.2 and Proposition 6.1. We omit the details.

Using the tools developed above, we can now easily prove other properties of the $H_n$'s. For example, we have the following two results.

*Proposition 6.2:* $H_{2n} = H_{2n-1} + H_{2n+1}$.

*Proof:* From (3.1), we can write directly

$$H_{2n-1} + H_{2n+1} = \sum_{s=0}^{n-1} \left|\begin{matrix} 2n - 1 - s \\ s \end{matrix}\right| 2^{n-1-s} + \sum_{r=0}^{n} \left|\begin{matrix} 2n + 1 - r \\ r \end{matrix}\right| 2^{n-r}$$

$$(6.1) \qquad = \left|\begin{matrix} 2n + 1 \\ 0 \end{matrix}\right| 2^n + \sum_{r=1}^{n} \left( \left|\begin{matrix} 2n - r \\ r - 1 \end{matrix}\right| + \left|\begin{matrix} 2n + 1 - r \\ r \end{matrix}\right| \right) 2^{n-r}.$$

Now let us observe that from the basic relation (1.1), it follows that

$$\left|\begin{matrix} 2n + 1 - r \\ r \end{matrix}\right| \equiv \left|\begin{matrix} 2n - r \\ r - 1 \end{matrix}\right| + \left|\begin{matrix} 2n - r \\ r \end{matrix}\right| \pmod 2.$$

So, by substitution of the right-hand side of this congruence into the coefficient of the large summand of equation (6.1), we obtain

$$\left|\begin{matrix} 2n - r \\ r - 1 \end{matrix}\right| + \left|\begin{matrix} 2n - r \\ r - 1 \end{matrix}\right| + \left|\begin{matrix} 2n - r \\ r \end{matrix}\right| \equiv \left|\begin{matrix} 2n - r \\ r \end{matrix}\right| \pmod 2.$$

Moreover, the coefficient of the first term can be trivially replaced by $\left|\begin{smallmatrix} 2n \\ 0 \end{smallmatrix}\right|$. Hence, we finally get

$$H_{2n-1} + H_{2n+1} = \sum_{r=0}^{n} \left|\begin{matrix} 2n - r \\ r \end{matrix}\right| 2^{n-r} = H_{2n}. \ \blacksquare$$

*Proposition 6.3:* $H_{2^h-2} + H_{2^h} = 2 \cdot H_{2^h+1}$.

*Proof:*

$$\begin{aligned} H_{2^h-2} + H_{2^h} &= (H_{2^h-3} + H_{2^h-1}) + (H_{2^h-1} + H_{2^h+1}) \quad \text{by Proposition 6.2} \\ &= (H_{2^h-3} + 2^{2^{h-1}}) + H_{2^h+1} \\ &= (H_1 \cdot 2^{2^{h-1}} + H_{2^h-3}) + H_{2^h+1} \\ &= H_{2^h+1} + H_{2^h+1} \quad \text{by Proposition 6.1.} \ \blacksquare \end{aligned}$$

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