# ADVANCED PROBLEMS AND SOLUTIONS 

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

## PROBLEMS PROPOSED IN THIS ISSUE

H-466 Proposed by Paul S. Bruckman, Edmonds, WA
Let $p$ be a prime of the form $a x^{2}+b y^{2}$, where $a$ and $b$ are relatively prime natural numbers neither of which is divisible by $p ; x$ and $y$ are integers. Prove that $x$ and $y$ are uniquely determined, except for trivial variations of sign.

H-467 Proposed by Larry Taylor, Rego Park, NY
Let $\left(a_{n}, b_{n}, c_{n}\right)$ be a primitive Pythagorean triple for $n=1,2,3,4$ where $a_{n}, b_{n}, c_{n}$ are positive integers and $b_{n}$ is even. Let $p \equiv 1$ (mod 8) be prime; $r^{2}+s^{2} \equiv t^{2}(\bmod p)$ where the Legendre symbol
$\left(\frac{(t+r) / 2}{p}\right)=1$.
Solve the following twelve simultaneous congruences:

$$
\begin{aligned}
&\left(a_{1}, b_{1}, c_{1}\right) \equiv(r, s, t), \\
&\left(a_{2}, b_{2}, c_{2}\right) \equiv(r, s,-t), \\
&\left(a_{3}, b_{3}, c_{3}\right) \equiv(s, r, t), \\
&\left(a_{4}, b_{4}, c_{4}\right) \equiv(s, r,-t) \quad(\bmod p) . \\
& \text { For example, if }(r, s, t) \equiv(3,4,5)(\bmod 17), \\
&\left(a_{1}, b_{1}, c_{1}\right)=(3,4,5) \\
&\left(a_{2}, b_{2}, c_{2}\right)=(105,208,233), \\
&\left(a_{3}, b_{3}, c_{3}\right)=(667,156,685), \\
&\left(a_{4}, b_{4}, c_{4}\right)=(21,20,29) .
\end{aligned}
$$

H-468 Proposed by Lawrence Somer, Washington, DC
Let $\left\{v_{n}\right\}_{n=0}^{\infty}$ be a Lucas sequence of the second kind satisfying the recursion relation

$$
v_{n+2}=a v_{n+1}+b v_{n},
$$

where $a$ and $b$ are positive odd integers and $v_{0}=2, v_{1}=a$. Show that $v_{2 n}$ has an odd prime divisor $p \equiv 3(\bmod 4)$ for $n \geq 1$. (This was proved by Sahib Singh
for the special case of the recurrence $\left\{L_{n}\right\}$ on page 136 of the paper "Thoro's Conjecture and Allied Divisibility Property of Lucas Numbers" in the April 1980 issue of The Fibonacci Quarterly.)

## SOLUTIONS

## A Triggy Problem

H-466 Proposed by J. A. Sjogren, U. of Santa Clara, Santa Clara, CA (Vol. 28, no. 4, November 1990)

Establish the following result:
Let $n$ be a whole number and, for any rational number $q$, let $[q$ ] be the greatest integer contained in $q$. Then

$$
f_{n}=\prod_{k=1}^{\left[\frac{n-1}{2}\right]}\left(3+2 \cos \frac{2 \pi k}{n}\right)
$$

Here, an empty product is to be interpreted as unity.
Solution by Paul S. Bruckman, Edmonds, WA
We consider the Chebychev polynomials of the second kind, defined as follows:

$$
\begin{equation*}
U_{n}(x)=\frac{a^{n+1}-b^{n+1}}{a-b}, \quad n=0,1,2, \ldots \tag{1}
\end{equation*}
$$

where

$$
\begin{equation*}
a=a(x)=x+\sqrt{x^{2}-1}, b=b(x)=x-\sqrt{x^{2}-1} \tag{2}
\end{equation*}
$$

It may be shown that $U_{n}(x)=0$ iff $x=\cos (\pi k /(n+1)), k=1,2, \ldots, n$. The $U_{n}(x)$ are polynomials of degree $n$, and their leading term is $(2 x)^{n}$. Therefore,

$$
\begin{equation*}
U_{n}(x)=2^{n} \prod_{k=1}^{n}\left(x-\cos \frac{k \pi}{n+1}\right) \tag{3}
\end{equation*}
$$

It follows that

$$
U_{n}(i x) U_{n}(-i x)=4^{n} \prod_{k=1}^{n}\left(x^{2}+\cos ^{2} \frac{k \pi}{n+1}\right)
$$

By a change in variable from $n$ to $n-1$ :

$$
\begin{equation*}
U_{n-1}(i x) U_{n-1}(-i x)=\prod_{k=1}^{n-1}\left(4 x^{2}+2+2 \cos \frac{2 k \pi}{n}\right) \tag{4}
\end{equation*}
$$

In particular, setting $x=1 / 2$, we obtain:

$$
\begin{equation*}
U_{n-1}(i / 2) U_{n-1}(-i / 2)=\prod_{k=1}^{n-1}\left(3+2 \cos \frac{2 k \pi}{n}\right) \tag{5}
\end{equation*}
$$

Next, using (1) and (2), we obtain

$$
a(i / 2)=\frac{1}{2} i(1 \pm \sqrt{5})=i \alpha \text { or } i \beta
$$

where $\alpha$ and $\beta$ are the usual Fibonacci constants; also

$$
b(i / 2)=\frac{1}{2} i(1 \mp \sqrt{5})=i \beta \text { or } i \alpha
$$

respectively. In either case,

$$
U_{n-1}(i / 2)=i^{n-1} F_{n} .
$$

Likewise,

$$
U_{n-1}(-i / 2)=(-i)^{n-1} F_{n}
$$

Consequently,

$$
\begin{equation*}
U_{n-1}(i / 2) U_{n-1}(-i / 2)=F_{n}^{2}, \quad n=1,2,3, \ldots \tag{6}
\end{equation*}
$$

Now, let $A_{n}$ denote the product expression indicated in the statement of the problem. For brevity, let $\theta_{k}=3+2 \cos (2 k \pi / n)$. Note that $\theta_{n-k}=\theta_{k}$. We consider two cases:

$$
\begin{aligned}
& \text { Case 1: } n=2 m \\
& \text { Then } A_{n}=\prod_{k=1}^{m-1} \theta_{k}=\prod_{k=m+1}^{n-1} \theta_{k} . \text { A1so, } \theta_{m}=1 \\
& \text { Hence, } A_{n}^{2}=\prod_{k=1}^{n-1} \theta_{k} .
\end{aligned}
$$

Case 2: $n=2 m+1$

$$
\text { Then } A_{n}=\prod_{k=1}^{m} \theta_{k}=\prod_{k=m+1}^{n-1} \theta_{k} \text {, so } A_{n}^{2}=\prod_{k=1}^{n-1} \theta_{k}
$$

In either case, we have

$$
\begin{equation*}
A_{n}^{2}=\prod_{k=1}^{n-1}\left(3+2 \cos \frac{2 k \pi}{n}\right) \tag{7}
\end{equation*}
$$

Comparing this last expression with (5) and (6), we see that

$$
\begin{equation*}
A_{n}^{2}=F_{n}^{2}, \quad n=1,2, \ldots \tag{8}
\end{equation*}
$$

Since $\theta_{k} \geq 1$, we see that $A_{n} \geq 1$. From this it follows that

$$
\begin{equation*}
A_{n}=F_{n}, \quad n=1,2, \ldots \text { Q.E.D. } \tag{9}
\end{equation*}
$$

Also solved by $S$. Rabinowitz and H.-J. Seiffert.

## Rings True

H-448 Proposed by T. V. Padmakumar, Trivandrum, South India (Vol. 28, no. 4, November 1990)

If $n$ is any number and $\alpha_{1}, a_{2}, \ldots, a_{m}$ are prime to $n\left(n>a_{1}, a_{2}, \ldots, a_{m}\right)$, then $\left(a_{1} a_{2} \ldots a_{m}\right)^{2} \equiv 1(\bmod n)$. [The number of positive integers less than $n$ and prime to it is denoted by $\phi(n)=m$.]

Solution by R. André-Jennin, Tunisia
Put $\mathbb{Z} / n \mathbb{Z}=\{\overline{0}, \overline{1}, \ldots,(n-1)\}$, and $U=\left\{\bar{a}_{1}, \bar{a}_{2}, \ldots, \bar{a}_{m}\right\}$. By hypothesis, $U$ is the multiplicative group of the invertible elements of the ring $\mathbb{Z} / n \mathbb{Z}$.

It is clear that the map $\bar{x} \rightarrow \bar{x}^{-1}$ is a one-to-one mapping of $U$ onto itself. Hence,

$$
\bar{a}_{i}^{-1}=\bar{a}_{\sigma(i)}, \text { for } i=1, \ldots, m
$$

where $\sigma$ is a permutation of $\{1,2, \ldots, m\}$.
Thus, in the ring $\mathbb{Z} / n \mathbb{Z}$,

$$
\left(\bar{a}_{1} \bar{a}_{2} \ldots \bar{a}_{m}\right)^{-1}=\bar{a}_{1}^{-1} \ldots \bar{a}_{m}^{-1}=\bar{a}_{\sigma(1)} \ldots \bar{a}_{\sigma(m)}=\bar{a}_{1} \ldots \bar{a}_{m}
$$

and so

$$
\left(\bar{a}_{1} \ldots \bar{a}_{m}\right)^{2}=\overline{1}
$$

or, in other words,

$$
\left(a_{1} \ldots a_{m}\right)^{2} \equiv 1 \quad(\bmod n)
$$

Also solved by D. Redmond and the proposer.

## A Recurrent Theme

H-449 Proposed by Ioan Sadoveanu, Ellensburg, WA (Vol. 29, no. 1, February 1991)

Let $G(x)=x^{k}+\alpha_{1} x^{k-1}+\cdots+\alpha_{k}$ be a polynomial with $c$ as a root of order p. If $G^{(p)}(x)$ denotes the $p^{\text {th }}$ derivative of $G(x)$, show that

$$
\begin{aligned}
& \left\{\frac{n^{p} c^{n-p}}{G^{(p)}(c)}\right\} \text { is a solution to the recurrence } \\
& u_{n}=c^{n-k}-a_{1} u_{n-1}-a_{2} u_{n-2}-\cdots-a_{k} u_{n-k}
\end{aligned}
$$

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY
The result is trivial if $c=0$, so we shall assume that $\mathcal{c} \neq 0$. Write

$$
G(x)=(x-c)^{p} H(x)
$$

where $H(c) \neq 0$. Let

$$
g(x)=x^{k} G(1 / x) \quad \text { and } \quad h(x)=x^{k-p_{H}}(1 / x)
$$

such that $g(x)=(1-c x)^{p} h(x)$, where $h(1 / c) \neq 0$. Denote the generating function of $u_{n}$ by $U(x)$. It is straightforward to check that

$$
g(x) U(x)=g(x)\left(\sum_{n=0}^{\infty} u_{n} x^{n}\right)=w_{1}(x)+\frac{x^{k}}{1-c x}
$$

for some polynomial $w_{1}(x)$ which depends on $u_{0}, u_{1}, \ldots, u_{k-1}$. Therefore,

$$
\begin{equation*}
U(x)=\frac{(1-c x) w_{1}(x)+x^{k}}{(1-c x) g(x)}=\frac{(1-c x) w_{1}(x)+x^{k}}{(1-c x)^{p+1} h(x)} . \tag{1}
\end{equation*}
$$

It follows that the characteristic equation for the recurrence is

$$
(x-c)^{p+1} H(x)=0
$$

Hence, $u_{n}=n^{p} c^{n}$ is a solution. We now proceed to improve this result.
There exist a polynomial $\omega_{2}(x)$ and constants $A_{1}, \ldots, A_{p+1}$ such that

$$
\begin{equation*}
U(x)=\frac{w_{2}(x)}{h(x)}+\frac{A_{1}}{1-c x}+\cdots+\frac{A_{p+1}}{(1-c x)^{p+1}} \tag{2}
\end{equation*}
$$

Consider

$$
\frac{1}{(1-c x)^{t}}=\sum_{n=0}^{\infty}\binom{t+n-1}{t-1}(c x)^{n}
$$

Since $\binom{t+n-1}{t-1}$ is a polynomial in $n$ of degree $t-1$, it is clear that

$$
A_{p+1} /(1-c x)^{p+1}
$$

is the only expansion in which the coefficient of $x^{n}$ contains $n^{p}$. Indeed, this coefficient is precisely $A_{p+1} n^{p} c^{n} / p!$. Equating the numerators in (1) and (2), we obtain

$$
(1-c x) w_{1}(x)+x^{k}=w_{2}(x)(1-c x)^{p+1}+h(x)\left\{A_{1}(1-c x)^{p}+\cdots+A_{p+1}\right\}
$$

Thus

$$
A_{p+1}=\frac{(1 / c)^{k}}{h(1 / c)}=\frac{1}{c^{p} H(c)}
$$

From the observation

$$
\begin{aligned}
(p+1)!H(c) & =\left.\frac{d^{p+1}}{d x^{p+1}}\left\{(x-c)^{p+1} H(x)\right\}\right|_{x=c}=\left.\frac{d^{p+1}}{d x^{p+1}}\{(x-c) G(x)\}\right|_{x=c} \\
& =(p+1) G^{(p)}(c),
\end{aligned}
$$

we conclude that

$$
u_{n}=\frac{A_{p+1} n^{p} c_{n}}{p!}=\frac{n^{p} c^{n-p}}{G^{(p)}(c)}
$$

is a solution to the given recurrence relation.
Also solved by P. Bruckman, R. André-Jeannin, and the proposer.

## Comparable

H-450 Proposed by R. André-Jeannin, Tunisia
(Vol. 29, no. 1, February 1991)
Compare the numbers

$$
\theta=\sum_{n=1}^{\infty} \frac{1}{F_{n}}
$$

and

$$
\theta^{\prime}=2+\sum_{n=1}^{\infty} \frac{1}{F_{n}\left(2 F_{n-1}^{2}+(-1)^{n-1}\right)\left(2 F_{n}^{2}+(-1)^{n}\right)}
$$

Solution by P. Bruckman, Edmonds, WA
We let

$$
\begin{align*}
& A_{n}=2 F_{n}^{2}+(-1)^{n}, \quad n=0,1,2, \ldots  \tag{1}\\
& D_{n}=F_{n} A_{n} A_{n-1}, \quad n=1,2,3, \ldots \tag{2}
\end{align*}
$$

We will prove the identity:

$$
\begin{equation*}
\frac{1}{D_{n}}=\frac{1}{F_{n}}+\frac{2 F_{n+1}}{A_{n}}-\frac{2 F_{n}}{A_{n-1}}, \quad n=1,2,3, \ldots \tag{3}
\end{equation*}
$$

The right member of (3) is equal to

$$
\frac{1}{D_{n}}\left[A_{n} A_{n-1}+2 F_{n}\left(F_{n+1} A_{n-1}-F_{n} A_{n}\right)\right]
$$

therefore, it suffices to prove the identity:

$$
\begin{equation*}
A_{n} A_{n-1}+2 F_{n}\left(F_{n+1} A_{n-1}-F_{n} A_{n}\right)=1, \quad n=1,2,3, \ldots . \tag{4}
\end{equation*}
$$

Let $S_{n}$ denote the left member of (4). We see from (1) that $A_{0}=A_{1}=1$; hence, $S_{1}=1$. It suffices to prove that $S_{n+1}-S_{n}=0, n=1,2, \ldots$, for this would imply (4). We first require some basic identities:

$$
\begin{equation*}
A_{n}=F_{n}^{2}+F_{n+1} F_{n-1}=F_{n+1}^{2}-F_{n} F_{n-1}=F_{n-1}^{2}+F_{n} F_{n+1} \tag{5}
\end{equation*}
$$

$$
\begin{align*}
& A_{n+1}-A_{n}=2 F_{n+1} F_{n}  \tag{6}\\
& A_{n+1}-A_{n-1}=2 F_{2 n}  \tag{7}\\
& F_{n+1} A_{n+1}+F_{n} A_{n-1}=A_{n}\left(F_{n+2}+2 F_{n}\right) \tag{8}
\end{align*}
$$

Proof of (5): Since $F_{n+1} F_{n-1}-F_{n}^{2}=(-1)^{n}$,

$$
\begin{aligned}
A_{n} & =2 F_{n}^{2}+F_{n+1} F_{n-1}-F_{n}^{2}=F_{n}^{2}+F_{n+1} F_{n-1}=\left(F_{n+1}-F_{n-1}\right)^{2}+F_{n+1} F_{n-1} \\
& =F_{n+1}^{2}-F_{n+1} F_{n-1}+F_{n-1}^{2}=F_{n+1}^{2}-F_{n-1}\left(F_{n+1}-F_{n-1}\right)=F_{n+1}^{2}-F_{n} F_{n-1} \\
& =F_{n-1}^{2}+F_{n+1}\left(F_{n+1}-F_{n-1}\right)=F_{n-1}^{2}+F_{n} F_{n+1} .
\end{aligned}
$$

Proof of (6): $A_{n+1}-A_{n}=F_{n+1}^{2}+F_{n+2} F_{n}-F_{n+1}^{2}+F_{n} F_{n-1}$
$=F_{n}\left(F_{n+2}+F_{n-1}\right)=F_{n}\left(F_{n+1}+F_{n}+F_{n+1}-F_{n}\right)=2 F_{n+1} F_{n}$.
Proof of (7): $A_{n+1}-A_{n-1}=F_{n+1}^{2}+F_{n+2} F_{n}-F_{n-1}^{2}-F_{n} F_{n-2}$
$=\left(F_{n+1}-F_{n-1}\right)\left(F_{n+1}+F_{n-1}\right)+F_{n}\left(F_{n+2}-F_{n-2}\right)$
$=F_{n} L_{n}+F_{n}\left(F_{n+1}+F_{n}-F_{n}+F_{n-1}\right)=2 F_{n} L_{n}=2 F_{2 n}$.
Proof of (8): $F_{n+1} A_{n+1}+F_{n} A_{n-1}=F_{n+1}\left(A_{n}+2 F_{n+1} F_{n}\right)+F_{n}\left(A_{n}-2 F_{n} F_{n-1}\right) \quad[$ by (6)]
$=\left(F_{n+1}+F_{n}\right) A_{n}+2 F_{n}\left(F_{n+1}^{2}-F_{n} F_{n-1}\right)=F_{n+2} A_{n}+2 F_{n} A_{n} \quad[$ using (5)]
$=\left(F_{n+2}+2 F_{n}\right) A_{n}$.
Therefore,
$S_{n+1}-S_{n}=A_{n+1} A_{n}+2 F_{n+1}\left(F_{n+2} A_{n}-F_{n+1} A_{n+1}\right)-A_{n} A_{n-1}-2 F_{n}\left(F_{n+1} A_{n-1}-F_{n} A_{n}\right)$
$=A_{n}\left(A_{n+1}-A_{n-1}\right)+2 A_{n}\left(F_{n}^{2}+F_{n+1} F_{n+2}\right)-2 F_{n+1}\left(F_{n+1} A_{n+1}+F_{n} A_{n-1}\right)$
$=A_{n}\left(2 F_{2 n}\right)+2 A_{n} A_{n+1}-2 F_{n+1} A_{n}\left(F_{n+2}+2 F_{n}\right)$ [using (5), (7), (8)]
$=2 A_{n}\left(F_{2 n}+A_{n+1}-F_{n+1}\left(F_{n+2}+2 F_{n}\right)\right)$
$=2 A_{n}\left(F_{2 n}+F_{n}^{2}+F_{n+1} F_{n+2}-F_{n+1} F_{n+2}-2 F_{n} F_{n+1}\right)=2 A_{n}\left(F_{2 n}-F_{n}\left(2 F_{n+1}-F_{n}\right)\right)$
$=2 A_{n}\left(F_{2 n}-F_{n}\left(F_{n+1}+F_{n-1}\right)\right)=2 A_{n}\left(F_{2 n}-F_{n} L_{n}\right)=0$.
This completes the proof of (4), and hence of (3).
We may now sum both sides of 93) over all natural numbers $n$, observing that all sums are absolutely convergent. The left sum is equal to

$$
\sum_{n=1}^{\infty} \frac{1}{D_{n}}=\theta^{\prime}-2
$$

Let $u_{n}=2 F_{n+1} / A_{n}$. The right sum is equal to

$$
\left[\sum_{n=1}^{\infty} \frac{1}{F_{n}}+u_{n}-u_{n-1}\right]=\sum_{n=1}^{\infty} \frac{1}{F_{n}}-u_{0}=\theta-2 \quad \begin{aligned}
& \text { (using the fact that } \\
& \left.u_{n} \rightarrow 0 \text { as } n \rightarrow \infty\right) .
\end{aligned}
$$

We conclude:

$$
\begin{equation*}
\theta^{\prime}=\theta \tag{9}
\end{equation*}
$$

Comment: This very interesting result furnishes us with a series equivalent to the much-studied series $\sum_{n=1}^{\infty} 1 / F_{n}$, but converging much more rapidly than the latter series. Thus,

$$
\theta=3+1 / 3+1 / 42+1 / 399+1 / 4655+1 / 50568+\cdots \doteq 3.3599
$$

