BOUNDS FOR THE CATALAN NUMBERS

A. V. Boyd
University of the Witwatersrand, Johannesburg 2050, South Africa
(Submitted May 1990)

1. Introduction

For the simple symmetric random walk on a two-dimensional lattice, it is well known (see, e.g., Feller [4], p. 361) that the probability of the origin begin revisited at the $2n^{\text{th}}$ step is

$$u_{2n} = 4^{-2n}\binom{2n}{n}^2, \quad (n = 0, 1, 2, \ldots);$$

and the Catalan number

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

(see Constantine [2], p. 61) is expressible as

$$c_n = \frac{2^{2n}}{n+1}\sqrt{u_{2n}}.$$

In a study of the transient behavior of the random walk Downham & Fotopoulos [3] have shown after much computation that

$$\frac{1}{\pi n}\left(1 - \frac{1}{4n} + \frac{1}{32n^2}\right) < u_{2n} < \frac{1}{\pi n}\left(1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{1}{32n^3}\right)$$

for $n = 1, 2, \ldots$, and this leads to inequalities for c_n which we strengthen by using standard analytical techniques. It is shown that, for $k \geq 3$ and every positive integer n,

$$1 + f(n, k) < \frac{1}{\pi n u_{2n}} < 1 + f(n, k) + \varepsilon_{k+1}$$

where

$$f(n, k) = \sum_{r=1}^{k} \frac{r!\binom{1/2}{r}^2}{n(n+1)\cdots(n+r-1)}$$

and

$$\varepsilon_{k+1} = \frac{(k-2)!}{4\pi(n+1)\cdot n(n+1)\cdots(n+k-1)}.$$

For any positive integer n,

$$\lim_{k\to\infty} \varepsilon_{k+1} = 0$$

and so both the bounds given by the inequalities tend to $1/\pi n u_{2n}$ as k increases; hence, u_{2n} can be approximated as accurately as desired.

Explicitly, for $k = 3$, the above results give

$$1 + \frac{1}{4n} + \frac{1}{32n(n+1)} + \frac{3}{128n(n+1)(n+2)} + \frac{1}{4\pi n(n+1)^2(n+2)}$$

$$> \frac{1}{\pi n u_{2n}} > 1 + \frac{1}{4n} + \frac{1}{32n(n+1)} + \frac{3}{128n(n+1)(n+2)}$$

and these are stronger than the inequalities of Downham & Fotopoulos.

2. Proof of the Inequalities

It is easily verified that

$$u_{2n} = \left\{\frac{\Gamma\left(n + \frac{1}{2}\right)}{n!\Gamma\left(\frac{1}{2}\right)}\right\}^2 = \frac{1}{\pi n} \Big/ \frac{\Gamma(n)\Gamma(n+1)}{\Gamma^2\left(n + \frac{1}{2}\right)}$$

and then, by Gauss's theorem (see Whittaker & Watson [5], p. 281):

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \text{ for } \operatorname{Re}(c - a - b) > 0,$$

it follows that

$$u_{2n} = 1/\pi n F\left(-\frac{1}{2}, -\frac{1}{2}; n; 1\right) \text{ since } n \text{ is a positive integer}$$

$$= 1/\pi n \left\{1 + \sum_{r=1}^{\infty} v_r\right\}$$

where

$$v_r = \frac{\left\{-\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \cdots \cdot \left(r - \frac{3}{2}\right)\right\}^2}{r!n(n+1) \cdots (n + r - 1)} = \frac{u_{2r-2}(r-1)!}{4rn(n+1) \cdots (n+r-1)}.$$

Since $v_r > 0$ ($r \geq 1$), it follows that $u_{2n} < 1/\pi n$ and so, if $r \geq 3$, then

$$v_r < \frac{(r-2)!/\pi}{4rn(n+1) \cdots (n+r-1)} < \frac{(r-3)!}{4\pi n(n+1) \cdots (n+r-1)};$$

hence, for $k \geq 4$,

$$\sum_{r=k}^{\infty} v_r < \frac{1}{4\pi} \sum_{r=k}^{\infty} \frac{(r-3)!}{n(n+1) \cdots (n+r-1)}$$

$$= \frac{(k-4)!}{4\pi n(n+1) \cdots (n+k-2)} \left\{\frac{k-3}{n+k-1} + \frac{(k-3)(k-2)}{(n+k-1)(n+k)} + \cdots\right\}$$

$$= \frac{(k-4)!}{4\pi n(n+1) \cdots (n+k-2)} \{F(k-3, 1; n+k-1; 1) - 1\}$$

$$= \frac{(k-4)!}{4\pi n(n+1) \cdots (n+k-2)} \left\{\frac{\Gamma(n+k-1)\Gamma(n+1)}{\Gamma(n+2)\Gamma(n+k-2)} - 1\right\}$$

by Gauss's theorem, since $n > -1$. This simplifies to

$$\sum_{r=k}^{\infty} v_r < \frac{(k-3)!}{(n+1) \cdot 4\pi n(n+1) \cdots (n+k-2)}.$$

From

$$1 + \sum_{r=1}^{k} v_r < 1 + \sum_{r=1}^{\infty} v_r < 1 + \sum_{r=1}^{k} v_r + \frac{(k-2)!}{(n+1) \cdot 4\pi n(n+1) \cdots (n+k-1)}$$

it then follows that, for $k \geq 3$,

$$1 + f(n, k) < \frac{1}{\pi n u_{2n}} < 1 + f(n, k) + \varepsilon_{k+1}$$

where

$$0 < \varepsilon_{k+1} \leq \frac{(k-2)!}{8\pi k!} = \frac{1}{8\pi(k-1)k} \to 0 \text{ as } k \to \infty.$$

3. Numerical Comparisons

The following table shows some bounds given in the cases $k = 3$ and $k = 4$ as well as the bounds obtained from the inequalities of Downham & Fotopoulos. For problems related to the computation of the integer c_n when n is large, see Campbell [1].

n		u_{2n}		c_n	
		Lower Bound	Upper Bound	Lower Bound	Upper Bound
1	D & F	.24868	.25863	.9974	1.0171
	$k = 3$	.24942	.25073	.9989	1.0015
	$k = 4$	.249778	.250429	.99956	1.00086
		$u_2 = .25$		$c_1 = 1$	
2	D & F	.140 504	.141 126	1.99914	2.00356
	$k = 3$	.140 560	.140 698	1.99954	2.00052
	$k = 4$	.140 605	.140 660	1.99986	2.00025
		$u_4 = .140\ 625$		$c_2 = 2$	
10	D & F	.031 045 161	.031 046 156	16795.935	16796.204
	$k = 3$	.031 045 315	.031 045 481	16795.977	16796.022
	$k = 4$	.031 045 390	.031 045 416	16795.997	16796.004
		$u_{20} = .031\ 045\ 401$		$c_{10} = 16796$	
100	D & F	.003 175 151 061	.003 175 151 160	$c_{100} \approx .896\ 5199 \times 10^{57}$	
	$k = 3$	.003 175 151 085	.003 175 151 088		
	$k = 4$	.003 175 151 086 636	.003 175 151 086 683		
		$u_{200} = .003\ 175\ 151\ 086\ 657$			

References

1. D. M. Campbell. "The Computation of Catalan Numbers." *Math. Magazine* 57 (1984):195-208.
2. G. M. Constantine. *Combinatorial Theory and Statistical Design*. New York: Wiley, 1987.
3. D. Y. Downham & S. B. Fotopoulos. "The Transient Behaviour of the Simple Random Walk in the Plane." *J. Appl. Prob.* 25 (1988):58-69.
4. W. Feller. *An Introduction to Probability Theory and Its Applications*. Vol. I, 3rd ed. New York: Wiley, 1968.
5. E. T. Whittaker & G. N. Watson. *A Course of Modern Analysis*. Cambridge, Mass.: Cambridge University Press, 1927.

*****