ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

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Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

Dedication. This year’s column is dedicated to Dr. A. P. Hillman in recognition of his 27 years of devoted service as editor of the Elementary Problems Section. Devotees of this column are invited to thank Abe by dedicating their next proposed problem to Dr. Hillman.

BASIC FORMULAS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$  

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-712 Proposed by Herta T. Freitag, Roanoke, VA

Prove that for all positive integers $n$, $\alpha(\sqrt{5}\alpha^n - L_{n+1})$ is a Lucas number.

B-713 Proposed by Herta T. Freitag, Roanoke, VA

Dedicated to Dr. A. P. Hillman

(a) Let $S$ be a set of three consecutive Fibonacci numbers. In a Pythagorean triple, $(a, b, c)$, $a$ is the product of the elements in $S$; $b$ is the product of two Fibonacci numbers (both larger than 1), one of them occurring in $S$; and $c$ is the sum of the squares of two members of $S$. Determine the Pythagorean triple and prove that the area of the corresponding Pythagorean triangle is the product of four consecutive Fibonacci numbers.

(b) Same problem as part (a) except that Fibonacci numbers are replaced by Lucas numbers.

B-714 Proposed by J. R. Goggins, Whiteinch, Glasgow, Scotland

Dedicated to Dr. A. P. Hillman

Define a sequence $G_n$ by $G_0 = 0, \quad G_1 = 4, \quad$ and $G_{n+2} = 3G_{n+1} - G_n - 2$ for $n \geq 0$. Express $G_n$ in terms of Fibonacci and/or Lucas numbers.
B-715 Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy
Dedicated to Dr. A. P. Hillman

Prove that if $s > 2$,
\[ F_m \equiv 0 \pmod{F_s^2} \text{ if and only if } m \equiv 0 \pmod{sF_s}. \]

B-716 Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA
Dedicated to Dr. A. P. Hillman

If $a$ and $b$ have the same parity, prove that $L_a + L_b$ cannot be a prime larger than 5.

B-717 Proposed by L. Kuipers, Sierre, Switzerland

Show that
\[ \arctan \frac{2}{5} = \sum_{n=0}^{\infty} \frac{(-1)^n L_{2n+1}}{2n+1} \cdot \frac{1}{2^{2n+1}}. \]

SOLUTIONS
Edited by A. P. Hillman

Differences in \{1, 2\}

B-688 Proposed by Russell Euler, Northwest Missouri State U., Maryville, MO

Find the number of increasing Euler sequences of integers such that 1 is the first term, $n$ is the last term, and the difference between successive terms is 1 or 2. [For example, if $n = 8$, then one such sequence is 1, 2, 3, 5, 6, 8 and another is 1, 3, 4, 6, 8.]

Solution by H.-J. Seiffert, Berlin, Germany

Let $A_n$ denote the set of all sequences having the desired properties. If $|A_n|$ denotes the number of elements of $A_n$, then clearly $|A_1| = |A_2| = 1$. We shall prove that $|A_n| = F_n$. This is true for $n = 1, 2$. Assume that it holds for all $k \in \{1, \ldots, n-1\}$ ($n \geq 3$). If $B_n$ and $C_n$ denote the set of all sequences of $A_n$, where the second term from the right equals $n - 2$ and $n - 1$, respectively, then we obviously have
\[ |B_n| = |A_{n-2}| \quad \text{and} \quad |C_n| = |A_{n-1}|. \]

Since $A_n = B_n \cup C_n$ and $B_n \cap C_n = \emptyset$, the induction hypothesis gives
\[ |A_n| = |B_n| + |C_n| = |A_{n-2}| + |A_{n-1}| = F_{n-2} + F_{n-1} = F_n. \]

This completes the induction proof.

Numbers with Even Zeckendorf Representations

B-689 Proposed by Philip L. Mana, Albuquerque, NM

Show that \( F_n^2 - 1 \) is a sum of Fibonacci numbers with distinct positive even subscripts for all integers \( n \geq 3 \).

Solution by Lawrence Somer, Washington, D.C.

It follows by identity \( I_{10} \) on page 56 of Hoggatt's *Fibonacci and Lucas Numbers* that
\[
F_n^2 - F_{n-2}^2 = F_{2n-2}.
\]
Thus,
\[
F_n^2 - 1 = F_{2n-2} + (F_{n-2}^2 - 1).
\]
The result now follows by induction upon noting that
\[
F_3^2 - 1 = 2^2 - 1 = 3 = F_4 \quad \text{and} \quad F_4^2 - 1 = 3^2 - 1 = 8 = F_6.
\]

Golden Geometric Progressions

B-689 Proposed by Herta T. Freitag, Roanoke, VA

Let \( S_k = a^{10k+1} + a^{10k+2} + a^{10k+3} + \ldots + a^{10k+10} \), where \( a = (1 + \sqrt{5})/2 \). Find positive integers \( b \) and \( c \) such that \( S_k/a^{10k+b} = c \) for all nonnegative integers \( k \).

Solution by Paul S. Bruckman, Edmonds, WA

\[
S_k = \sum_{i=1}^{10} a^{10k+i} = a^{10k} \left( \frac{a^{11} - a}{a - 1} \right) = \frac{a^{10k+1}}{a^1 - 1}(a^{10} - 1)
\]
\[
= a^{10k+2} \cdot a^5(a^5 + b^5) = 11a^{10k+7}.
\]
We see that the values \( b = 7 \), \( c = 11 \) solve the problem.


Rectangles in Similar Rectangles

B-691 Proposed by Heiko Harborth, Technische U. Braunschweig, W. Germany

Herta T. Freitag asked whether a golden rectangle can be inscribed into a larger golden rectangle (all four vertices of the smaller are points on the sides of the larger one). An answer follows from the solution of the generalized problem: Which rectangles can be inscribed into larger similar rectangles?

Solution by Russell Jay Hendel, Dowling College, Oakdale, NY

All nonsquare rectangles can be inscribed in larger similar rectangles.
Indeed, suppose a given rectangle, \( R_1 \), has sides \( a \) and \( b \) with \( a \neq b \). Set 
\[ c = \frac{\max(a, b)}{\min(a, b)} \]
and consider the rectangle, \( R_2 \), with sides \( \max(a, b) \) and \( c \).

\( R_2 \) is similar to \( R_1 \), because
\[ \frac{c}{\max(a, b)} = \frac{\max(a, b)}{\min(a, b)}, \]
\( R_1 \) can be inscribed in \( R_2 \), with common side \( \max(a, b) \), because \( c \geq \min(a, b) \), and \( R_2 \) is larger than \( R_1 \) because \( c > \min(a, b) \) when \( a \neq b \). This completes the proof.

**Editor's Note:** The proposer made the tacit assumption that all the vertices of the smaller rectangle are interior points of the sides of the larger one. With this assumption, Paul S. Bruckman, Herta T. Freitag, and the proposer showed that the rectangles have to be squares.

**A Fibonacci Factorization**

**B-692 Proposed by Gregory Wulczyn, Lewisburg, PA**

Let

\[ G(a, b, c) = -4 + L_{2a}^2 + L_{2b}^2 + L_{2c}^2 + L_{2a}L_{2b}L_{2c}. \]

Prove or disprove that each of \( F_{a+b+c}, F_{b+c-a}, F_{c+a-b}, \) and \( F_{a+b+c} \) is an integral divisor of \( G(a, b, c) \) for all odd positive integers \( a, b, \) and \( c \).

**Solution by Russell Jay Hendel, Dowling College, Oakdale, NY**

We prove the stronger assertion
\[ 25F_{a+b+c}^2 - 25F_{a-b}^2 - 25F_{a-b+c}^2 - 25F_{a+b-c}^2 = L_{2a}L_{2b}L_{2c} + L_{2a}^2 + L_{2b}^2 + L_{2c}^2 - 4. \]

The proof is verbatim identical to the published solution to B-669, Vol. 29, no. 2, p. 185, with, however, the word odd replaced by even.

"From the identity
\[ L_{2m+n}F_{m-n} = L_{2m} - (-1)^{m+n}L_{2n} \]
we get [setting \((-1)^{a+b+c} = e\)]
\[ 25F_{a+b+c}^2 - 25F_{a-b}^2 - 25F_{a-b+c}^2 - 25F_{a+b-c}^2 = L_{2a}L_{2b}L_{2c} - eL_{2a}^2 + eL_{2b}^2 - eL_{2c}^2 \]
\[ = L_{2c}[L_{2a}L_{2b} + L_{2a-2b}] - eL_{2a}^2 - eL_{2a}^2 + eL_{2a-2b}L_{2a-2b} \]
\[ = L_{2a}[L_{2a}L_{2b}] - eL_{2a}^2 - e[L_{4a} + L_{4b}] \]
\[ = L_{2a}L_{2b}L_{2c} - e[L_{2a}^2 + L_{2b}^2 + L_{2c}^2 - 4], \]
and for \( a, b, \) and \( c \) odd (actually for \( a + b + c \) odd), the given identity is established."

**Also solved by Paul S. Bruckman, Norbert Jensen, Bob Prielipp, and the proposer.**

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A Combinatorial Problem

B-693 Proposed by Daniel C. Fielder & Cecil O. Alford, Georgia Tech, Atlanta, GA

Let \( A \) consist of all pairs \( \{x, y\} \) chosen from \( \{1, 2, \ldots, 2n\} \), \( B \) consist of all pairs from \( \{1, 2, \ldots, n\} \), and \( C \) of all pairs from \( \{n + 1, n + 2, \ldots, 2n\} \). Let \( S \) consist of all sets \( T = \{P_1, P_2, \ldots, P_k\} \) with the \( P_i \) (distinct) pairs in \( A \). How many of the \( T \) in \( S \) satisfy at least one of the the conditions:

(i) \( P_i \cap P_j = \emptyset \) for some \( i \) and \( j \), with \( i \neq j \),
(ii) \( P_i \in B \) for some \( i \), or
(iii) \( P_i \in C \) for some \( i \)?

Solution by Philip L. Mana, Albuquerque, NM

Let \( C(n, k) \) denote \( \binom{n}{k} \). There are \( C(2n, 2) \) pairs in \( A \); thus \( C(C(2n, 2), k) \) sets \( T \) in \( S \). The sets \( T \) meeting none of the conditions (i), (ii), (iii) can be written in the form

\[
U = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_k, y_k\}\}
\]

with the \( x_\ell \) a set of \( k \) integers chosen from \( \{1, 2, \ldots, n\} \) and satisfying

\[
x_1 < x_2 < \ldots < x_k
\]

and the \( y_\ell \) a permutation of \( k \) distinct integers from \( \{n + 1, n + 2, \ldots, 2n\} \). The number of such sets \( U \) is

\[
C(n, k)P(n, k) = k!(\frac{n}{k})^2;
\]

hence, the number of sets \( T \) satisfying at least one of the conditions is

\[
C(C(2n, 2), k) - C(n, k)P(n, k).
\]

Also solved by the proposers, who indicated that the problem arose in a combinatorial study in parallel processing.

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