ACCELERATION OF THE SUM OF FIBONACCI RECIPROCALs

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The topic of Aitken acceleration (sometimes called "Aitken's $\Delta^2$ process") appears in many numerical analysis texts but is usually confined to the solution of equations by fixed-point iteration. (Interesting examples of this occur in [4] and [6], wherein $x = 1 + 1/x$, equivalent to the characteristic equation of the Fibonacci difference relation, is solved by iteration.) Conspicuous in suggesting its applicability in other contexts are [1], [2], [5], and [7].

Briefly, a convergent sequence $x_1, x_2, \ldots, x_n$ with limit $x$ is amenable to acceleration if the ratio of consecutive errors is approximately constant $(x - x_n)/(x - x_{n-1}) \sim r$. It follows that $r$ is approximately the ratio of consecutive differences $r \sim (x_n - x_{n-1})/(x_{n-1} - x_{n-2})$. Substituting this value of $r$ into the approximation for the ratio of errors, solving for $x$, and relabeling $x$ as $x_n^*$ yields the more rapidly converging sequence

$$x_n^* = x_n - (x_n - x_{n-1})^2/(x_n - 2x_{n-1} + x_{n-2}) = x_n - (\Delta_n)^2/\Delta_n^2,$$

where the second form uses the $\Delta$ notation for first and second differences,

$$\Delta_n = x_n - x_{n-1} \quad \text{and} \quad \Delta_n^2 = \Delta_n - \Delta_{n-1}.$$

An occasionally mentioned use other than functional iteration is the accelerated convergence of Taylor series [5], often possible because of the behavior of the error term. A trivial but revealing example of this is the geometric series; acceleration of any three consecutive partial sums takes us directly to the limit since the ratio of errors (or differences) is, in fact, exactly constant.

Because of its intriguing resemblance to a geometric series, the sum of Fibonacci reciprocals provides a dramatic illustration of both the increase in speed of convergence attainable and the rarely mentioned possibilities of repeated acceleration. To be sure, in 1972 Broussard achieved at least 83-digit accuracy in

$$S = \sum_{j=1}^{\infty} 1/F_j = 3.3598856662\ldots$$

by evaluating $S_{400} = \sum_{j=1}^{400} 1/F_j$ to 400 digits (see [3] for an extensive bibliography), but this need not detract from what can be learned by pursuing this example.

Although $S_7 = 5047/1560 = 3.235\ldots$ has a relative error of 4%, it and the six previous partial sums themselves can be accelerated with pencil-and-paper arithmetic to produce $1391/414 = 3.359903\ldots$, with relative error only .0005%. Accelerating the information provided by the first seven terms has reduced the inaccuracy in our estimate of $S$ by a factor of 7000.

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1992] 179
ACCELERATION OF THE SUM OF FIBONACCI RECIPROCAL

Observe that the ratios of consecutive differences are close to $1/a$, $-1/a^2$, and $1/a^3$ in the three stages of acceleration, where $a = (\sqrt{5} + 1)/2 = 1.618\ldots$.

An explanation for this lies in the Binet formula for the $j$th Fibonacci number

$$F_j = a^j(1 - (-1/a^2)^j)/\sqrt{5},$$

from which

$$1/F_j = \frac{\sqrt{5}}{a^j} \sum_{k=0}^{m} (-1/a^2)^j.$$ 

Thus, a partial sum is given by

$$S_n = \sum_{j=1}^{n} 1/F_j = \sqrt{5} \sum_{k=0}^{m} \sum_{j=1}^{n} (-1)^{j+k} / a(2k+1);$$

and the error, or tail, is the double sum

$$S - S_n = \sqrt{5} \sum_{k=0}^{m} \sum_{j=n+1}^{m} (-1)^{j+k} / a(2k+1);$$

$$= \sqrt{5} \sum_{k=0}^{m} \frac{(-1)^{k+n}}{(a(2k+1) - (-1)^k)(a^2k+1)}.$$ 

Each stage of the acceleration will eliminate the currently dominant component in the error term, in this case successively peeling off those of order $(1/a)^n$, $(-1/a^3)^n$, $(1/a^5)^n$, etc. [7]. Generally, if the error in a sequence is $\sum a_i^2$, with $1 > |a_1| > |a_2| > \ldots$, then acceleration changes the error by removing the $a_1$ term, altering the coefficients of the other $a_i^n$, and possibly introducing new terms of order

$$\prod_j (a_j/a_1)^{k_1}a_i^n < a_i^n.$$ 

In the Fibonacci case the orders of the new terms happen to coincide with those of terms already present. Acton [1] points out that when convergence is near, roundoff error can be amplified by the process, causing later accelerations to wander off the mark.

Another possibility for approximating $S$ is to correct the partial sums $S_n$ by estimating their tails as $a/F_n$, making use of the well-known asymptotic $F_n + a/F_n = \alpha F_n$ (the inherent "geometric" character of the Fibonacci reciprocals). For example, $S_7 + a/F_7 = 3.35972\ldots$ is already quite good, and repeated acceleration of the first seven such corrected terms produces $3.35988567$, competitive with four accelerations of the first nine partial sums themselves, the correction being equivalent to starting with one acceleration already achieved.

Incidentally, passing to the limit in the expression for $S_n$ gives an alternative formula for $S$ itself, apparently not widely known:

$$S = \sum_{j=1}^{\infty} 1/F_j = \sqrt{5} \sum_{k=0}^{m} \frac{(-1)^k}{a(2k+1) - (-1)^k};$$

$$= \sqrt{5}[1/(a - 1) - 1/(a^3 + 1) + 1/(a^5 - 1) - \ldots].$$ 

And yes, this too can be accelerated, about as well as $S_n + a/F_n$!

**Acknowledgment**

I would like to thank the referee for some helpful comments, including the observation that this final expression for $S$ can also be derived by applying formula 4.17 from [3], with $q = -1$ and $\beta = -1/a$. 

180 [May
ACCELERATION OF THE SUM OF FIBONACCI RECIPROCALS

References


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