ACCELERATION OF THE SUM OF FIBONACCI RECIPROCALS

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The topic of Aitken acceleration (sometimes called "Aitken's Δ^2 process") appears in many numerical analysis texts but is usually confined to the solution of equations by fixed-point iteration. (Interesting examples of this occur in [4] and [6], wherein $x = 1 + 1/x$, equivalent to the characteristic equation of the Fibonacci difference relation, is solved by iteration.) Conspicuous in suggesting its applicability in other contexts are [1], [2], [5], and [7].

Briefly, a convergent sequence $x_1, x_2, \ldots, x_n$ with limit x is amenable to acceleration if the ratio of consecutive errors is approximately constant $(x - x_n)/(x - x_{n-1}) \sim r$. It follows that r is approximately the ratio of consecutive differences $r \sim (x_n - x_{n-1})/(x_{n-1} - x_{n-2})$. Substituting this value of r into the approximation for the ratio of errors, solving for x, and relabeling x as x_n^* yields the more rapidly converging sequence

$$x_n^* = x_n - (x_n - x_{n-1})^2/(x_n - 2x_{n-1} + x_{n-2}) = x_n - (\Delta_n)^2/\Delta_n^2,$$

where the second form uses the Δ notation for first and second differences,

$$\Delta_n = x_n - x_{n-1} \quad \text{and} \quad \Delta_n^2 = \Delta_n - \Delta_{n-1}.$$

An occasionally mentioned use other than functional iteration is the accelerated convergence of Taylor series [5], often possible because of the behavior of the error term. A trivial but revealing example of this is the geometric series; acceleration of any three consecutive partial sums takes us directly to the limit since the ratio of errors (or differences) is, in fact, exactly constant.

Because of its intriguing resemblance to a geometric series, the sum of Fibonacci reciprocals provides a dramatic illustration of both the increase in speed of convergence attainable and the rarely mentioned possibilities of repeated acceleration. To be sure, in 1972 Brousseau achieved at least 83-digit accuracy in

$$S = \sum_{j=1}^{\infty} 1/F_j = 3.3598856662\ldots$$

by evaluating $S_{400} = \sum_{j=1}^{400} 1/F_j$ to 400 digits (see [3] for an extensive bibliography), but this need not detract from what can be learned by pursuing this example.

Although $S_7 = 5047/1560 = 3.235\ldots$ has a relative error of 4%, it and the six previous partial sums themselves can be accelerated with pencil-and-paper arithmetic to produce $1391/414 = 3.359903\ldots$, with relative error only .0005%. Accelerating the information provided by the first seven terms has reduced the inaccuracy in our estimate of S by a factor of 7000.

$x_n = S_n$	Δ_n	Δ_n^2	x_n^*	Δ_n^*	Δ_n^{2*}	x_n^{**}	Δ_n^{**}	Δ_n^{2**}	x_n^{***}
1									
2	1/1								
5/2	1/2	-1/2	3						
17/6	1/3	-1/6	7/2	1/2					
91/30	1/5	-2/15	10/3	-1/6	-2/3	27/8			
279/120	1/8	-3/40	101/30	1/30	1/5	121/36	-1/72		
5047/1560	1/13	-5/104	403/120	-1/120	-1/24	84/25	-1/900	23/1800	1391/414

Observe that the ratios of consecutive differences are close to $1/\alpha$, $-1/\alpha^3$, and $1/\alpha^5$ in the three stages of acceleration, where $\alpha = (\sqrt{5} + 1)/2 = 1.618\ldots$.

An explanation for this lies in the Binet formula for the j^{th} Fibonacci number

$$F_j = \alpha^j(1 - (-1/\alpha^2)^j)/\sqrt{5},$$

from which

$$1/F_j = \frac{\sqrt{5}}{\alpha^j}\sum_{k=0}^{\infty}(-1/\alpha^2)^{jk}.$$

Thus, a partial sum is given by

$$S_n = \sum_{j=1}^{n} 1/F_j = \sqrt{5}\sum_{k=0}^{\infty}\sum_{j=1}^{n}(-1)^{jk}/\alpha^{(2k+1)j}$$

and the error, or tail, is the double sum

$$S - S_n = \sqrt{5}\sum_{k=0}^{\infty}\sum_{j=n+1}^{\infty}(-1)^{jk}/\alpha^{(2k+1)j} = \sqrt{5}\sum_{k=0}^{\infty}\frac{(-1)^{k+kn}}{(\alpha^{2k+1} - (-1)^k)(\alpha^{2k+1})^n}.$$

Each stage of the acceleration will eliminate the currently dominant component of the error term, in this case successively peeling off those of order $(1/\alpha)^n$, $(-1/\alpha^3)^n$, $(1/\alpha^5)^n$, etc. [7]. Generally, if the error in a sequence is $\sum c_i a_i^n$, with $1 > |a_1| > |a_2| > \cdots$, then acceleration changes the error by removing the a_1 term, altering the coefficients of the other a_i^n, and possibly introducing new terms of order

$$\left[\prod_j (a_j/a_1)^{k_j} a_i\right]^n < a_2^n.$$

In the Fibonacci case the orders of the new terms happen to coincide with those of terms already present. Acton [1] points out that when convergence is near, roundoff error can be amplified by the process, causing later accelerations to wander off the mark.

Another possibility for approximating S is to correct the partial sums S_n by estimating their tails as α/F_n, making use of the well-known asymptotic $F_{n+j} \sim \alpha^j F_n$ (the inherent "geometric" character of the Fibonacci reciprocals). For example, $S_7 + \alpha/F_7 = 3.35972\ldots$ is already quite good, and repeated acceleration of the first seven such corrected terms produces 3.35988567, competitive with four accelerations of the first nine partial sums themselves, the correction being equivalent to starting with one acceleration already achieved.

Incidentally, passing to the limit in the expression for S_n gives an alternative formula for S itself, apparently not widely known:

$$S = \sum_{j=1}^{\infty} 1/F_j = \sqrt{5}\sum_{k=0}^{\infty}\frac{(-1)^k}{\alpha^{2k+1} - (-1)^k}$$

$$= \sqrt{5}[1/(\alpha - 1) - 1/(\alpha^3 + 1) + 1/(\alpha^5 - 1) - \cdots].$$

And yes, this too can be accelerated, about as well as $S_n + \alpha/F_n$!

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Referemces

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