SCHUR FUNCTIONS AND FIBONACCI IDENTITIES

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1. Introduction

In this article we use the elementary theory of symmetric functions and the theory of characters of representations of the symmetric group to derive identities involving generalized Fibonacci and Lucas numbers. Not all the identities obtained are new; what is possibly of greater interest is the approach, which may lead to further results. We have included some preparatory material on partitions, Schur functions and characters in Sections 2, 3, and 5. Proofs of the statements made there may be found, among many other places, in [1] and [2]. Character calculations similar to those carried out in this paper are found in [3].

Let a and b be any two unequal complex numbers. Define the Lucasian pairs $\{U_n\}$ and $\{V_n\}$ by

$$U_n = \frac{a^n - b^n}{a - b}, \quad V_n = a^n + b^n; \quad n = 0, 1, 2, \ldots .$$

Then U_n and V_n satisfy the recurrences

$$U_{n+2} = PU_{n+1} - QU_n, \quad V_{n+2} = PV_{n+1} - QV_n,$$

where $P = a + b$, $Q = ab$, $P^2 - 4Q \neq 0$. In case $P = 1$, $Q = -1$, put $U_n = F_n$, $V_n = L_n$. Then F_n and L_n are the Fibonacci and Lucas numbers, respectively.

Let $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_k$ be positive integers. One of our basic identities has the form

$$(1.1) \qquad V_{\rho_1} V_{\rho_2} \ldots V_{\rho_k} = \sum_{j=0}^{\left[\frac{n}{2}\right]} A_{\rho_1, \rho_2, \ldots, \rho_k; j} U_{n-2j+1}$$

where the A's are simply expressible in terms of Q and certain characters of the symmetric group. An identity inverse to (1.1) is also obtained. For certain choices of $\{\rho_1, \rho_2, \ldots, \rho_k\}$, the relevant characters can be fairly readily computed. In this way we obtain, for instance, the identity

$$(1.2) \qquad \sum_{j=0}^{\left[\frac{m}{2}\right]} \left[\binom{m - 2^q + 1}{j} - \binom{m - 2^q + 1}{j - 2^q}\right] Q^j U_{m-2j+1} = P^{m-2^q+1} U_{2^q}.$$

In Section 7 we use a different approach to derive identities involving Lucas numbers and certain generalized binomial coefficients.

2. Partitions and Tableaux

A partition is a finite sequence of nonnegative integers:

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$$

in nonincreasing order. A part of λ is a nonzero member of $\{\lambda_1, \lambda_2, \ldots, \lambda_t\}$. The number of parts is the length, $\ell(\lambda)$, of λ. The sum $|\lambda| = \lambda_1, \lambda_2, \ldots, \lambda_k$, where $k = \ell(\lambda)$ is the weight of λ. λ is said to be a partition of $|\lambda|$. Occasionally we use an "exponential" notation for λ:

$$\lambda = 1^{\beta_1} 2^{\beta_2} \ldots m^{\beta_m}.$$

Here, β_i is the number of times i occurs in the sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$.

$[\lambda]$, the diagram of λ, is the set of all points (i, j) in Z^2 such that $1 \le j \le \lambda_i$. Thus, the diagram of (3, 3, 2, 1) is

$$\begin{matrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{matrix}$$

Sometimes it is convenient to use squares rather than dots. Let λ and μ be partitions with $|\tau| = |\mu|$. A semi-standard tableau of shape λ and content μ is an arrangement of μ_1 1's, μ_2 2's, μ_3 3's, etc., in the squares of the diagram of λ so that the rows are nondecreasing and the columns are strictly increasing. For example, the semi-standard tableaux of shape (4, 2) and content (3, 2, 1) are

1	1	1	2
2	3		

and

1	1	1	3
2	2		

Figure 1

Partitions may be ordered lexicographically. That is,

$\lambda > \mu$ if $\lambda_1 > \mu_1$ or if $\lambda_1 = \mu_1$ and $\lambda_2 > \mu_2$

or if $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$, and $\lambda_3 > \mu_3$, etc.

Semi-standard tableaux of shape λ and content μ can exist only if $\lambda \ge \mu$. (This condition is not sufficient.)

3. Schur Functions

We shall be working in the ring $Z[x_1, x_2, \ldots, x_n]$ of polynomials in n independent variables with integer coefficients. Such a polynomial is symmetric if it is invariant under all permutations of the variables. For each n-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in N^n, we denote by x^α the monomial

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x^{\alpha_n}.$$

If λ is a partition of length $\le n$, the polynomial

$$m_\lambda(x_1, x_2, \ldots, x_n) = \sum x^\alpha,$$

where the summation is over all permutations α of $\{\lambda_1, \lambda_2, \ldots, \lambda_i\}$ is symmetric. The power sums

$$p_r = \sum_{i=1}^{n} x_i^r$$

are symmetric, as are the products

$$p_\rho = p_{\rho_1} p_{\rho_2} \ldots p_{\rho_k} \quad (\rho = (\rho_1, \rho_2, \ldots, \rho_k))$$

With every partition λ we can associate another type of symmetric function, called a Schur function, or S-function. Let λ be a partition $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and put $\delta = (n-1, n-2, \ldots, 1, 0)$. Define

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j + n - j}), \ 1 \le i \le n, \ 1 \le j \le n.$$

Then

$$a_\delta = \det(x_i^{m-j}) = \prod_{1 \le i < j \le m} (x_i - x_j)$$

is the Vandermonde determinant. Clearly, a_δ divides $a_{\lambda+\delta}$. The quotient

$$s_\lambda = s_\lambda(x_1, x_2, \ldots, x_m) = \frac{a_{\lambda+\delta}}{a_\delta}$$

is a symmetric homogeneous polynomial of degree $|\lambda|$ which is called a Schur function.

The sets $M_m = \{m_\lambda \mid \ell(\lambda) \le m\}$ and $S = \{s_\lambda \mid \ell(\lambda) \le m\}$ are Z-bases for Λ_m, the set of symmetric polynomials in m variables with coefficients in Z. Thus, for example, we may express the polynomials s_λ as integral linear combinations of the polynomials m_μ. We have

$$(3.1) \qquad s_\lambda = \sum_{|\mu| = |\lambda|} K_{\lambda,\mu} m_\mu.$$

It is possible to show that the Kostka number $K_{\lambda,\mu}$ is the number of semi-standard tableaux of shape λ and content μ. Therefore, $K_{\lambda,\mu}$ is a nonnegative integer that vanishes if $\lambda < \mu$.

To express the polynomials p_ρ as integral linear combinations of Schur functions, we require the characters of $\sum_m$, the symmetric group on m letters. We have

$$(3.2) \qquad p_\rho = \sum_{|\lambda| = |\rho|} \chi_\rho^\lambda s_\lambda,$$

where χ_ρ^λ is the character of the irreducible representation of $\sum_m$ determined by λ evaluated at the conjugate class of $\sum_m$ consisting of permutations with cycle-partition ρ.

Inverse to (3.2) is the relation

$$(3.3) \qquad s_\lambda = \frac{1}{m!} \sum_{|\rho| = |\lambda|} c_\rho \chi_\rho^\lambda p_\rho$$

where c_ρ is the number of permutations with cycle-partition ρ; i.e.,

$$c_\rho = \frac{m!}{1^{\gamma_1} 2^{\gamma_2} \ldots m^{\gamma_m} (\gamma_1)! (\gamma_2)! \ldots (\gamma_m)!}$$

with $\rho = 1^{\gamma_1} 2^{\gamma_2} \ldots m^{\gamma_m}$ and $|\rho| = m$.

4. Basic Identities

If there are only two independent variables x_1 and x_2, and if $\ell(\mu) \ge 3$, then $m_\mu = 0$. In this case (3.1) may be put in the form

$$(4.1) \qquad s_\lambda(x_1, x_2) = \sum_{k=0}^{\left[\frac{n}{2}\right]} K_{\lambda,(k,n-k)} m_{(k,n-k)}(x_1, x_2),$$

where $n = |\lambda|$. There can be no semi-standard tableau of shape λ and content μ if $\ell(\lambda) > \ell(\mu)$ because each of the $\ell(\lambda)$ rows of the tableau must be headed by a distinct integer chosen from a set of $\ell(\mu)$ integers. Thus, the only nontrivial case of (4.1) occurs when $\ell(\lambda) \le 2$. In this case it is not hard to see that, if $0 \le j \le \left[\frac{n}{2}\right]$ and $0 \le k \le \left[\frac{n}{2}\right]$, we have

$$K_{(j,n-j),(k,n-k)} = 1 \text{ if } k \ge j, \text{ and}$$

$$K_{(j,n-j),(k,n-k)} = 0 \text{ if } k < j,$$

whence

$$s(j,\ n-j)(x_1,\ x_2) = \sum_{k=j}^{\left[\frac{n}{2}\right]} m_{(k,\,n-k)}(x_1,\ x_2)$$
$$= x_1^j x_2^j (x_1^{n-2j} + x_1^{n-2j-1}x_2 + \cdots + x_2^{n-2j})$$

or

$$s_{(j,\,n-j)}(x_1,\ x_2) = \frac{(x_1x_2)\ (x_1^{n-2j+1} - x_2^{n-2j+1})}{x_1 - x_2}. \tag{4.2}$$

With U_n and V_n defined as in the introduction and $\rho = (\rho_1,\ \rho_2,\ \ldots,\ \rho_n)$, put

$$V_\rho = V_{\rho_1} V_{\rho_2}\ \ldots\ V_{\rho_k}. \tag{4.3}$$

Then, from (4.2), we have

$$s_{(j,\,n-j)}(a,\ b) = Q^j U_{n-2j+1}. \tag{4.4}$$

Moreover,

$$p_r(a,\ b) = V_r$$

so that, with $|\rho| = n$, (3.2) becomes

$$V_\rho = \sum_{j=0}^{\left[\frac{n}{2}\right]} \chi_\rho^{(j,\,n-j)} Q^j U_{n-2j+1}, \tag{4.5}$$

our first basic identity. For example, in the Fibonacci case, taking $\rho = (5,\ 3,\ 2)$ and referring to the table of characters of Σ_{10} in [1], we have

$$\begin{aligned}(-1)\ \chi_{(5,\,3,\,2)}^{(j,\,10-j)} F_{11-2j} &= F_{11} - (-1)F_9 + F_7 - 0 \cdot F_5 + (-1)F_3 - 2F_1\\ &= 89 + 34 + 13 - 0 - 2 - 2 = 132\\ &= 11 \cdot 4 \cdot 3 = L_5L_3L_2 = L_{(5,\,3,\,2)}.\end{aligned}$$

From (3.3) we get our second basic identity

$$\begin{cases} Q^j n! U_{n-2j+1} = \displaystyle\sum_{|\rho| = n} c_\rho \chi_\rho^{(j,\,n-j)} V_\rho, & \text{where } 0 \le j \le \dfrac{n}{2} \\ 0 = \displaystyle\sum_{|\rho| = n} c_\rho \chi_\rho^\lambda V_\rho, & \text{if } \ell(\lambda) \ge 3. \end{cases} \tag{4.6}$$

5. Special Cases of the First Basic Identity

In some cases it is not difficult to compute $\chi_\rho^{(j,\,n-j)}$. We use the Murnaghan-Nakayama Rule, which permits an inductive calculation. This requires some preliminary explanation.

Let $(i,\ j)$ be the point in the i^{th} row (counting downward) and j^{th} column (counting to the right) in $[\rho]$, the diagram of ρ. The hook $H_{i,j}^\rho$ consists of the point $(i,\ j)$ together with the points of $[\rho]$ directly to its right and directly below. The number of points in $H_{i,j}^\rho$, the length of the hook, is denoted by $h_{i,j}^\rho$. The points $(k,\ j)$, $k > i$, form the leg of $H_{i,j}^\rho$. The number of points in the leg of $H_{i,j}^\rho$ is called the leg-length and is denoted by $\ell_{i,j}^\rho$. The point of $H_{i,j}^\rho$ furthest to the right of $(i,\ j)$ is called the hand of the hook, while the point of $H_{i,j}^\rho$ furthest below $(i,\ j)$ is called its foot. To $H_{i,j}^\rho$ corresponds a portion of the rim of $[\rho]$ which is of the same length. It consists of the points on the rim between the hand and the foot. To $H_{1,2}^{5,3,1}$, for example, there correspond the encircled points of $[5,\ 3,\ 1]$ as follows:

$$\begin{array}{ccccc} \cdot & \cdot & \odot & \odot & \odot \\ \cdot & \odot & \odot & & \\ \cdot & & & & \end{array}$$

The associated part of the rim, $R^{\rho}_{i,j}$, is called a rim-hook. It is important to notice that the result

$$[\rho]\backslash R^{\rho}_{i,j}$$

of removing $R^{\rho}_{i,j}$ from $[\rho]$ is again the diagram of a partition; e.g.,

$$[5, 3, 1]\backslash R^{(5,3,1)}_{1,2} = \begin{array}{ll} \bullet & \bullet \\ \bullet & \\ \bullet & \end{array} = [2, 1^2].$$

The Murnaghan-Nakayama Rule is the following: Let λ and ρ be partitions of m, with

$$\rho = (1^{\beta}, 2^{\beta_2} \ \ldots \ k^{\beta_k} \ \ldots \ m^{\beta_m}).$$

Suppose $\beta_k \geq 1$ and let

$$\pi = (1^{\beta_1} \, 2^{\beta_2} \ \ldots \ k^{\beta_k - 1} \ \ldots \ m^{\beta_m}).$$

Then

$$\chi^{\lambda}_{\rho} = \sum_{\substack{i,\, j \\ h^{\lambda}_{ij} = k}} (-1)^{\rho^{\lambda}_{ij}} \chi^{\lambda\backslash R^{\lambda}_{ij}}_{\pi}. \tag{5.1}$$

Thus, by removing one occurrence of k from ρ and all k rim-hooks from λ, we can express χ^{λ}_{ρ} in terms of characters of lower order. Repeated application of this procedure allows us to compute χ^{λ}_{ρ} for any λ and ρ.

Let us assume that $j \leq m/2$. In case $\rho = (r, 1^{m-r})$ we can compute $\chi^{(j,\, m-j)}_{\rho}$ inductively by removing 1-hooks from $[j, m-j]$. The Murnaghan-Nakayama Rule yields

$$\begin{cases} \chi^{(j,\, m-j)}_{(r,\, 1^{m-r})} = \chi^{(j-1,\, m-j)}_{r,\, 1^{m-r-1}} + \chi^{(j,\, m-j-1)}_{r,\, 1^{m-r-1}} & \text{if } j < \dfrac{m}{2} \\[2ex] \chi^{(j,\, j)}_{(r,\, 1^{m-r})} = \chi^{(j-1,\, j)}_{r,\, 1^{m-r-1}} & \text{if } j = \dfrac{m}{2} \end{cases} \tag{5.2}$$

Note the resemblance between (5.2) and the binomial recurrence. It is not hard to show, using induction on m, that

$$\chi^{(j,\, m-j)}_{(r,\, 1^{m-r})} = \binom{m-r}{j} - \binom{m-r}{j-1} + \binom{m-r}{j-r} - \binom{m-r}{j-r-1}. \tag{5.3}$$

If $r = 1$, (5.3) becomes

$$\chi^{(j,\, m-j)}_{1^m} = \binom{m-1}{j} - \binom{m-1}{j-2} \tag{5.4}$$

(**Remark**: (5.4) may also be obtained from the Frame-Robinson-Thrall formula for the degree of an irreducible representation of Σ_n.)

When $r = 2$, (5.3) can be written

$$\chi^{(j,\, m-j)}_{2,\, 1^{m-2}} = \binom{m-3}{j} - \binom{m-3}{j-4}. \tag{5.5}$$

Using the same method as that used to establish (5.3), we can show that

$$\begin{aligned} \chi^{(j,\, m-j)}_{r,\, s,\, 1^{m-r-s}} = {} & \binom{m-r-s}{j} - \binom{m-r-s}{j-1} + \binom{m-r-s}{j-r} - \binom{m-r-s}{j-r-1} \\ & + \binom{m-r-s}{j-s} - \binom{m-r-s}{j-s-1} \\ & + \binom{m-r-s}{j-r-s} - \binom{m-r-s}{j-r-s-1}. \end{aligned} \tag{5.6}$$

If $s = 2$, we have

$$(5.7) \qquad \chi^{(j,\,m-j)}_{r,\,2,\,1^{m-r-2}} = \binom{m-r-3}{j} - \binom{m-r-3}{j-4} + \binom{m-r-3}{j-r} - \binom{m-r-3}{j-r-4}$$

and if, in addition, $r = 4$, then

$$(5.8) \qquad \chi^{(j,\,m-j)}_{4,\,2,\,1^{m-6}} = \binom{m-7}{j} - \binom{m-7}{j-8}.$$

Each of (5.3) through (5.8) yields, via (4.5), a Fibonacci identity. We have, for example,

$$(5.4)' \qquad \sum_{j=0}^{\left[\frac{m}{2}\right]} Q^j\left[\binom{m-1}{j} - \binom{m-1}{j-2}\right]U_{m-2j+1} = V_1^m = P^{m-1}U_2,$$

$$(5.5)' \qquad \sum_{j=0}^{\left[\frac{m}{2}\right]} Q^j\left[\binom{m-3}{j} - \binom{m-3}{j-4}\right]U_{m-2j+1} = V_1^{m-2}V_2 = P^{m-3}U_4,$$

$$(5.8)' \qquad \sum_{j=0}^{\left[\frac{m}{2}\right]} Q^j\left[\binom{m-7}{j} - \binom{m-7}{j-8}\right]U_{m-2j+1} = V_1^{m-6}V_2V_4 = P^{m-7}U_8.$$

An expression similar to (5.6) but involving 2^{q+1} binomial coefficients may be given for

$$\chi^{(j,\,m-j)}_{t_1,\,t_2,\,\ldots,\,t_q,\,1^{m-(t_1+\cdots+t_q)}}$$

In case $t_i = 2^i$, $1 \le i \le q - 1$, this expression may be simplified to give the expected generalization of (5.4), (5.5) and (5.8):

$$(5.9) \qquad \chi^{(j,\,m-j)}_{2,\,4,\,8,\,\ldots,\,2^{q-1},\,1^{m-2q+2}} = \binom{m-2+1}{j} - \binom{m-2+1}{j-2},$$

yielding the Fibonacci identity

$$(5.9)' \qquad \sum_{j=0}^{\left[\frac{m}{2}\right]} Q^j\left[\binom{m-2^q+1}{j} - \binom{m-2^q+1}{j-2^q}\right]U_{m-2j+1} = V_1^{m-2^q+2}V_2V_4 \cdots V_{2^q}$$
$$= P^{m-2^q+1}U_{2^q}.$$

If we reason similarly with rectangular partitions, i.e., partitions of the form t^k we obtain, from (4.5), the formulas

$$V_t^k = \sum^{\frac{k-1}{2}} \binom{k}{i} Q^{it} V_{(k-2i)t} \qquad k \text{ odd},$$

and

$$V_t^k = \sum^{\frac{k}{2}-1} \binom{k}{i} Q^{it} V_{(k-2i)t} + \binom{k}{\frac{1}{2}k} Q^{\frac{tk}{2}} \qquad k \text{ even}.$$

However, these identities are well known and not especially difficult to prove directly (see [4]).

6. Special Cases of the Second Basic Identity

If $\lambda = (n)$, then χ^λ_ρ is the identity character and (4.6) gives

$$(6.1) \qquad \sum_{|\rho| = n} c_\rho V_\rho = n!\,U_{n+1}.$$

If $\lambda = 1^n$, then $\chi_\rho^\lambda = \varepsilon(\rho)$, the alternating character. That is, $\varepsilon(\rho) = 1$ if the permutations with cycle-partition type ρ are even and $\varepsilon(\rho) = -1$ if these permutations are odd. From (4.6) we deduce

$$(6.2) \qquad \sum_{|\rho| = n} c_\rho \varepsilon_\rho V_\rho = 0 \quad \text{if } n \geq 3.$$

If $\lambda = (1, n - 1)$, then χ^λ is the so-called "natural" character and $\chi_\rho^\lambda = \gamma_1 - 1$ where $\rho = 1^{\gamma_1} 2^{\gamma_2} \ldots n^{\gamma_n}$. In other words, χ_ρ^λ is one less than the number of elements left fixed by permutations with cycle-partition ρ. From (4.6) we have

$$\sum_{|\rho| = n} c_\rho (\gamma_1(\rho) - 1) V_\rho = Qn! U_{n-1}$$

which, in conjunction with (6.1) gives

$$\sum_{|\rho| = n} c(\rho)\gamma_1(\rho) L_\rho = \sum_{|\rho| = n} c_\rho V_\rho + Qn! U_{n-1} = n!(U_{n+1} + QU_{n-1})$$

or, finally,

$$(6.3) \qquad \sum_{|\rho| = n} c_\rho \gamma_1(\rho) V_\rho = n! PU_n = n! V_1 U_n.$$

Lastly, if $\lambda = (2, 1^{n-2})$, then χ^λ is the character conjugate to the natural character, i.e.,

$$\chi_\rho^{2, 1^{n-2}} = \varepsilon(\rho)(\gamma_1(\rho) - 1).$$

Then, (4.6) yields, using (6.2),

$$(6.4) \qquad \sum_{|\rho| = n} c_\rho \varepsilon(\rho) \gamma_1(\rho) V_\rho = 0 \quad \text{if } n \geq 4.$$

The following chart illustrates (6.1) through (6.4) for $n = 4$ in the Fibonacci case.

ρ	c_ρ	$\varepsilon(\rho)$	$\gamma_1(\rho)$	L_ρ	$c_\rho L_\rho$	$c_\rho \varepsilon_\rho L_\rho$	$c_\rho \gamma_1(\rho) L_\rho$	$c_\rho \varepsilon(\rho) \gamma_1(\rho) L_\rho$
1	1	1	4	1	1	1	4	4
21	6	-1	2	3	18	-18	36	-36
2	3	1	0	9	27	27	0	0
31	8	1	1	4	32	32	32	32
4	6	-1	0	7	42	-42	0	0
		sums			$120 = 4!F_5$	0	$72 = 4!F_4$	0

A Generalization

Using a different approach, we generalize the identities established in Section 6. First, several additional concepts will be introduced.

Let

$$\rho = 1^{\gamma_1} 2^{\gamma_2} \ldots n^{\gamma_n} \quad \text{and} \quad \sigma = 1^{\beta_1} 2^{\beta_2} \ldots n^{\beta_n}$$

be partitions. We define the "generalized binomial coefficient" $\binom{\rho}{\sigma}$ by

$$(7.1) \qquad \binom{\rho}{\sigma} = \binom{\gamma_1}{\beta_1}\binom{\gamma_2}{\beta_2} \cdots \binom{\gamma_n}{\beta_n}$$

when the quantities on the right are ordinary binomial coefficients. $\binom{\rho}{\sigma}$ is itself an ordinary binomial coefficient when ρ and σ are suitable rectangular partitions. Clearly $\binom{\rho}{\sigma} = 0$ if $\gamma_i < \beta_i$ for some i, $1 \leq i \leq n$.

If $\gamma_i \geq \beta_i$, $1 \leq i \leq n$, we define the partition $\rho - \sigma$ by

$$(7.2) \qquad \rho - \sigma = 1^{\gamma_1 - \beta_1} 2^{\gamma_2 - \beta_2} \ldots n^{\gamma_n - \beta_n}.$$

Let

$$(7.3) \qquad z_\rho = \frac{n!}{c_\rho} = 1^{\gamma_1} 2^{\gamma_2} \ \ldots \ n^{\gamma_n} \gamma_1! \gamma_2! \ \ldots \ \gamma_n!$$

(z_ρ is the order of the centralizer of a permutation of cycle-type ρ). It is easy to show that

$$(7.4) \qquad \binom{\rho}{\sigma} = \frac{z_\rho}{z_\sigma z_{\rho - \sigma}}$$

whenever $\rho - \sigma$ is defined.

The r^{th} elementary symmetric function $e_r(x_1, \ldots, x_m)$ is the sum of all products of r distinct variables x_i so that $e_0 = 1$ and

$$e_r = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le m} x_{i_1} x_{i_2} \ \ldots \ x_{i_r}.$$

The r^{th} *complete* symmetric function $h_r(x_1, \ldots, x_m)$ is the sum of all monomials of total degree r, so that, for example,

$$h_3(x_1, x_2, \ldots, x_m) = x_1^3 + x_2^3 + \cdots + x_1^2 x_2 + \cdots + x_1 x_2 x_3 + \cdots .$$

In particular, $h_0 = 1$ and $h_1 = e_1$. For $r < 0$, it is convenient to put $h_r = e_r = 0$.

Our generalizations of the results of Section 6 are based on the identities

$$(7.5) \qquad \sum_{|\rho| = n} \frac{\binom{\rho}{\sigma}}{z_\rho} p_\rho = \frac{p_\sigma h_{n - |\sigma|}}{z_\sigma}$$

and

$$(7.6) \qquad \sum_{|\rho| = n} \frac{\varepsilon_\rho \binom{\rho}{\sigma}}{z_\rho} p_\rho = \frac{\varepsilon_\sigma p_\sigma e_{n - |\sigma|}}{z_\sigma}.$$

We prove only (7.5); the proof of (7.6) is similar.

Our proof of (7.5) is based on (7.4) and the identity

$$(7.7) \qquad \sum_{|\rho| = n} \frac{p_\rho}{z_\rho} = h_n.$$

(For a proof of (7.7), see [2], p. 17.)

Noting that $p \ = p \ p$, we have

$$\sum_{|\rho| = n} \frac{\binom{\rho}{\sigma}}{z_\rho} p_\rho = p_\sigma \sum_{|\rho| = n} \frac{\binom{\rho}{\sigma}}{z_\rho} p_{\rho - \sigma} = \frac{p_\sigma}{z_\sigma} \sum_{|\rho| = n} \frac{p_{\rho - \sigma}}{z_{\rho - \sigma}} = \frac{p_\sigma}{z_\sigma} \sum_{|\tau| = n - |\sigma|} \frac{p_\tau}{z_\tau}$$

$$= \frac{p_\sigma}{z_\sigma} h_{n - |\sigma|},$$

thus proving (7.5).

Observing that

$$h_r(a, b) = a_1^r + a^{r-1} b + \cdots + b^r = \frac{a^{r+1} - b^{r+1}}{a - b} = U_{r+1},$$

we find, on putting $x_1 = a_1$, $x_2 = b_2$, $x_3 = x_4 = \cdots = 0$ in (7.5), the identity

$$(7.8) \qquad \sum_{|\rho| = n} \frac{\binom{\rho}{\sigma} V_\rho}{z_\rho} = \frac{V_\sigma}{z_\sigma} U_{n - |\sigma| + 1},$$

which, using (7.3), can be written

$$(7.9) \qquad \sum_{|\rho| = n} c_\rho \binom{\rho}{\sigma} V_\rho = \frac{n! c_\sigma V_\sigma U_{n - |\sigma| + 1}}{|\sigma|!}.$$

Likewise, since

$$e_1(a, b, 0, \ldots) = a + b = P,$$
$$e_2(a, b, 0, \ldots) = ab = Q,$$
$$e_r(a, b, 0, \ldots) = 0, \text{ if } r \geq 3,$$

we obtain from (7.6),

$$(7.10) \qquad \sum_{|\rho| = n} \varepsilon_\rho \binom{\rho}{\sigma} c_\rho V_\rho = \frac{\varepsilon_\sigma c_\sigma n! P V_\sigma}{|\sigma|}$$

if $|\sigma| = n - 1$,

$$(7.11) \qquad \sum_{|\rho| = n} \varepsilon_\rho \binom{\rho}{\sigma} c_\rho V_\rho = \frac{\varepsilon_\sigma c_\sigma n! Q V_\sigma}{|\sigma|}$$

if $|\sigma| \leq n - 2$, and

$$(7.12) \qquad \sum_{|\rho| = n} \varepsilon_\rho \binom{\rho}{\sigma} c_\rho V_\rho = 0$$

if $|\sigma| \leq n - 3$.

If we specialize σ to be a partition of length 1, i.e., $\sigma = k^1$, then $\binom{\rho}{\sigma} = \gamma_k(\rho)$, $\varepsilon_\sigma = (-1)^{k-1}$, $c_\sigma = (k - 1)!$, and (7.9), (7.10), (7.11), and (7.12) yield

$$(7.13) \qquad \sum_{|\rho| = n} c_\rho \gamma_k(\rho) V_\rho = \frac{n! V_k U_{n-k+1}}{k},$$

$$(7.14) \qquad \sum_{|\rho| = n} \varepsilon_\rho c_\rho \gamma_k(\rho) V_\rho = \frac{(-1)^n n! P V_k}{k} \text{ if } k = n - 1,$$

$$(7.15) \qquad \sum_{|\rho| = n} \varepsilon_\rho c_\rho \gamma_k(\rho) V_\rho = \frac{(-1)^{n-1} n! Q V_k}{k} \text{ if } k = n - 2,$$

and

$$(7.16) \qquad \sum_{|\rho| = n} \varepsilon_\rho c_\rho \gamma_k(\rho) V_\rho = 0 \text{ if } k \leq n - 3,$$

which are generalizations of (6.3) and (6.4).

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