FIBONACCI SEQUENCES IN FINITE GROUPS

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0. Introduction

The Fibonacci sequence and its related higher-order sequences (tribonacci, quatranacci, k-nacci) are generally viewed as sequences of integers. In 1960, Wall [4] considered Fibonacci sequences modulo some fixed integer m; i.e., Fibonacci sequences of elements of $\mathbb{Z}_m$. He proved that these sequences were periodic for any m. Shah [3] partially determined for which integers the Fibonacci sequence modulo m contained the complete residue system, $\mathbb{Z}_m$. The papers of Wall [4] and Shah [3] provided the motivation for Wilcox's [5] study of the Fibonacci sequence in finite abelian groups.

This paper is in the spirit of [3], [4], and [5]. It addresses not only the traditional Fibonacci (2-nacci) sequence, but also the k-nacci sequence, and does so for finite (not necessarily abelian) groups.

1. Definitions and Notation

A *k-nacci sequence in a finite group* is a sequence of group elements x_0, x_1, x_2, x_3, ..., x_n, ... for which, given an initial (seed) set x_0, ..., x_{j-1}, each element is defined by

$$x_n = \begin{cases} x_0 x_1 \ldots x_{n-1} & \text{for } j \le n < k \\ x_{n-k} x_{n-k+1} \ldots x_{n-1} & \text{for } n \ge k \end{cases}.$$

We also require that the initial elements of the sequence, x_0, ..., x_{j-1}, generate the group, thus forcing the k-nacci sequence to reflect the structure of the group. The k-nacci sequence of a group G seeded by x_0, ..., x_{j-1} is denoted by $F_k(G;\ x_0, \ldots, x_{j-1})$.

The classic Fibonacci sequence in the integers modulo m can be written as $F_2(\mathbb{Z}_m;\ 0,\ 1)$. We call a 2-nacci sequence of group elements a *Fibonacci sequence of a finite group*.

A finite group G is *k-nacci sequenceable* if there exists a k-nacci sequence of G such that every element of the group appears in the sequence.

A sequence of group elements is *periodic* if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the *period* of the sequence. The sequence a, b, c, d, b, c, d, b, c, d, ... is periodic after the initial element a and has period 3. We denote the period of a k-nacci sequence $F_k(G;\ x_0, \ldots, x_{j-1})$ by $P_k(G;\ x_0, \ldots, x_{j-1})$. A sequence is *simply periodic* with period k if the first k elements in the sequence form a repeating subsequence. For example, a, b, c, d, e, a, b, c, d, e, ... is simply periodic with period 5.

2. Theorems

Theorem 1: A k-nacci sequence in a finite group is simply periodic.

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Proof: Let n be the order of G. Since there are n^k distinct k-tuples of elements of G, at least one of the k-tuples appears twice in a k-nacci sequence of G. Therefore, the subsequence following this k-tuple repeats; hence, the k-nacci sequence is periodic.

Since the sequence is periodic, there exist natural numbers i and j, with $i > j$, such that

$$x_{i+1} = x_{j+1},\ x_{i+2} = x_{j+2},\ x_{i+3} = x_{j+3},\ \ldots,\ x_{i+k} = x_{j+k}.$$

By the defining relation of a k-nacci sequence, we know that

$$x_i = x_{i+k}(x_{i+(k-1)})^{-1}(x_{i+(k-2)})^{-1} \ldots (x_{i+1})^{-1}$$

and

$$x_j = x_{j+k}(x_{j+(k-1)})^{-1}(x_{j+(k-2)})^{-1} \ldots (x_{j+1})^{-1}.$$

Hence, $x_i = x_j$, and it then follows that

$$x_{i-1} = x_{j-1},\ x_{i-2} = x_{j-2},\ \ldots,\ x_{i-j} = x_{j-j} = x_0.$$

Therefore, the sequence is simply periodic.

This is a generalization of a theorem of Wall [4], which states that $F(\mathbb{Z}_m;\ 0,\ 1)$, the classically seeded Fibonacci sequence of the integers modulo m, is simply periodic. From the proof of Theorem 1, we have $|G|^k$ as an upper bound for the period of any k-nacci sequence in a group G.

We will now address the periods of k-nacci sequences in specific classes of groups. A Group D_n is *dihedral* if

$$D_n = \langle a,\ b : a^n = b^2 = e \quad\text{and}\quad ba = a^{-1}b\rangle.$$

The order of the group D_n is $2n$. Note that in a dihedral group generated by a and b,

$$(ab)^2 = abab = aa^{-1}b^2 = e \text{ and } (ba)^2 = baba = baa^{-1}b = e.$$

Theorem 2: Consider the dihedral group D_n for some $n \geq 3$ with generators a, b. Then $P_k(D_n;\ a,\ b) = P_k(D_n;\ b,\ a) = 2k + 2$.

Proof: Let the orders of a and b be n and 2, respectively. If $k = 2$, the possible sequences are

$$a,\ b,\ ab,\ a^{-1},\ a^2b,\ ab,\ a,\ b,\ \ldots$$

and

$$b,\ a,\ a^{-1}b,\ b,\ a^{-1},\ ab,\ b,\ a,\ \ldots,$$

both of which have period 6. If $k \geq 3$, the first k elements of $F_k(D_n;\ a,\ b)$ are

$$x_0 = a,\ x_1 = b,\ x_2 = ab,\ x_3 = (ab)^2,\ \ldots,\ x_{k-1} = (ab)^{2^{k-3}}.$$

This sequence reduces to

$$a,\ b,\ ab,\ e,\ e,\ \ldots,\ e,\ e$$

where $x_j = e$ for $3 \leq j \leq k - 1$. Thus,

$$x_k = \prod_{i=0}^{k-1} x_i = abab = e, \qquad x_{k+1} = \prod_{i=1}^{k} x_i = bab = a^{-1},$$

$$x_{k+2} = \prod_{i=2}^{k+1} x_i = aba^{-1} = a^2b, \qquad x_{k+3} = \prod_{i=3}^{k+2} x_i = a^{-1}a^2b = ab,$$

$$x_{k+4} = \prod_{i=4}^{k+3} x_i = a^{-1}a^2bab = e.$$

It follows that $x_{k+j} = e$ for $4 \leq j \leq k$. We also have:

$$x_{k+k+1} = \prod_{i=k+1}^{k+k} x_i = a^{-1}a^2bab = e, \qquad x_{k+k+2} = \prod_{i=k+2}^{k+k+1} x_i = a^2bab = a,$$

$$x_{k+k+3} = \prod_{i=k+3}^{k+k+2} x_i = aba = b, \qquad x_{k+k+4} = \prod_{i=k+4}^{k+k+3} x_i = ab.$$

Since the elements succeeding x_{2k+2}, x_{2k+3}, x_{2k+4}, depend on a, b, and ab for their values, the cycle begins again with the $2k+2^{\text{nd}}$ element; i.e., $x_0 = x_{2k+2}$. Thus, the period of $F_k(D_n;\, a,\, b)$ is $2k + 2$. If we choose to seed the sequence with the generators in the other order, we see that the sequence b, a, ba, $(ba)^2$, $(ba)^4$, $(ba)^8$, ..., $(ba)^{2^{k-3}}$ reduces to b, a, ba, e, e, ..., e, e and the proof works similarly.

If a group is generated by i elements, then it is said to be an *i-generated group*.

Theorem 3: If G is a 2-generated group with generators a and b, and the identity element appears in $F_2(G;\, a,\, b)$ or $F_2(G;\, b,\, a)$, a Fibonacci sequence of G, then G is abelian.

Proof: Without loss of generality consider the sequence $F_2(G;\, a,\, b)$ and suppose the identity, e, is the $n+1^{\text{st}}$ element of this Fibonacci sequence for some natural number n. The n^{th} element of the sequence may be any element of the group. Thus, we have a sequence

$$a,\ b,\ \ldots,\ s,\ e,\ \ldots\ .$$

What precedes s? Only s^{-1} could satisfy the defining relation for the $n-1^{\text{st}}$ position. Similarly, s^2 must be in the $n-2^{\text{nd}}$ sequence position, s^{-3} in the $n-3^{\text{rd}}$, and so on, forming the sequence

$$a,\ b,\ \ldots,\ s^{-8},\ s^5,\ s^{-3},\ s^2,\ s^{-1},\ s^1,\ e,\ \ldots\ .$$

Since these elements have exponents generated using the relation $u_{i-2} = -u_{i-1} + u_i$, which is equivalent to $u_i = u_{i-1} + u_{i-2}$, we find the Fibonacci sequence of integers occurring in the exponents of s, with alternating signs. Hence, a Fibonacci sequence of the group has one of two forms:

(i) n odd: The sequence is

$$s^{u_n},\ s^{-u_{n-1}},\ s^{u_{n-2}},\ \ldots,\ s^5,\ s^{-3},\ s^2,\ s^{-1},\ s^1,\ e.$$

In this case, we have

$$s^{u_n} = a,\ s^{-u_{n-1}} = b$$

(which implies $s^{u_{n-1}} = b^{-1}$), and $s^{u_{n-2}} = ab$. Since

$$s^{u_{n-1}}\, s^{u_{n-2}} = s^{u_{n-1} + u_{n-2}} = s^{u_n},$$

we have $b^{-1}ab = a$, or $ab = ba$. Therefore, the group is abelian.

(ii) n even: The sequence is

$$s^{-u_n},\ s^{u_{n-1}},\ s^{-u_{n-2}},\ \ldots,\ s^5,\ s^{-3},\ s^2,\ s^{-1},\ s^1,\ e.$$

In this case, we have

$$s^{-u_n} = a,\ s^{u_{n-1}} = b$$

(which implies $s^{-u_{n-1}} = b^{-1}$), and $s^{-u_{n-2}} = ab$. Since

$$s^{-u_{n-1}}\, s^{-u_{n-2}} = s^{-(u_{n-1} + u_{n-2})} = s^{-u_n},$$

we have $b^{-1}ab = a$, or $ab = ba$. Therefore, the group is abelian.

The converse of Theorem 3 does not hold. Consider the abelian group

$$A = \langle a,\ b\colon a^9 = b^2 = e \text{ and } ba = ab\rangle.$$

The Fibonacci sequences of this group are:

$$a,\ b,\ ab,\ a,\ a^2b,\ a^3b,\ a^5,\ a^8b,\ a^4b,\ a^3,\ a^7b,\ ab,\ a^8,\ b,\ a^8b,\ a^8,\ a^7b,\ a^6b,\ a^4,\ ab,\ a^5b,\ a^6,\ a^2b,\ a^8b,\ a,\ b,\ ab,\ \ldots,$$

and

$$b,\ a,\ ab,\ a^2b,\ a^3,\ a^5b,\ a^8b,\ a^4,\ a^3b,\ a^7b,\ a,\ a^8b,\ b,\ a^8,\ a^8b,\ a^7b,\ a^6,\ a^4b,\ ab,\ a^5,\ a^6b,\ a^2b,\ a^8,\ ab,\ b,\ a,\ ab,\ \ldots\ .$$

The elements e, a^2, and a^7 do not appear in either sequence.

Corollary: A 2-nacci sequenceable group is cyclic.

Proof: Let G be a 2-nacci sequenceable group. Then G is either 1- or 2-generated. If G is 2-generated, then since e appears in the 2-nacci sequence of G, we can construct the sequence in terms of an element $s \in G$ as in the proof of Theorem 3. Every element of G appears in its 2-nacci sequence, and therefore all the elements of G may be represented in terms of a single element, s. Hence, G is 1-generated, or cyclic.

For $k \geq 3$, k-nacci sequenceable groups are not, in general, abelian. The dihedral group of six elements is 3-nacci sequenceable.

Theorem 4: If the identity element appears in a Fibonacci sequence of a 2-generated group, then the collection of subscripts of the sequence elements x_i for which $x_i = e$ contains a sequence which has an arithmetic progression.

Proof: By Theorem 3 the group $G = \langle a, b\rangle$ is abelian. Hence, the n^{th} term of the sequence has the form $a^{u_{n-1}}b^{u_n}$. By a theorem of Wall [4], we know that the terms where $u_n \equiv 0 \pmod{m}$ have subscripts that form a simple arithmetic progression. Thus, the sequences of elements $a, a, a^2, \ldots, a^{u_n}$ and $b, b, b^2, b^3, \ldots, b^{u_n}$ both have e occurring in positions whose subscripts form arithmetic progressions, with the period of the occurrence of e depending on the order of a and b. The period of this induced occurrence of e in $a, b, ab, ab^2, a^2b^3, \ldots$ will be the least common multiple of the period of e in $a, a, a^2, \ldots$ and the period of e in $b, b, b^2, b^3, \ldots$. Hence, the positions of e in $a, b, ab, ab^2, a^2b^3, \ldots$ will have subscripts which contain an arithmetic progression.

3. An Open Question

It is clear that a homomorphic image of a k-nacci sequenceable group is k-nacci sequenceable. The extension of a k-nacci sequenceable group by a k-nacci sequenceable group is not necessarily k-nacci sequenceable. In fact, the direct product of k-nacci sequenceable groups is not necessarily k-nacci sequenceable.

We refer to the abelian group

$$A = \langle a, b\colon a^9 = b^2 = e \text{ and } ba = ab\rangle.$$

The group $\langle b\rangle$ has a Fibonacci sequence

$$F_2(\langle b\rangle;\ e,\ b) = e,\ b,\ b,\ e,\ \ldots,$$

and hence is 2-nacci sequenceable. The group $\langle a\rangle$ has a sequence

$$F_2(\langle a\rangle;\ e,\ a) = e,\ a,\ a,\ a^2,\ a^3,\ a^5,\ a^8,\ a^4,\ a^3,\ a^7,\ a,\ a^8,\ e,\ a^8,\ a^8,\ a^7,\ a^6,\ a^4,\ a,\ a^5,\ a^6,\ a^2,\ a^8,\ a,\ e,\ a,\ a,\ \ldots$$

and hence is 2-nacci sequenceable. We have already seen that A, the direct product of $\langle a\rangle$ and $\langle b\rangle$, is not 2-nacci sequenceable.

Question: Are all nonsimple k-nacci sequenceable groups nontrivial extensions of a k-nacci sequenceable group by a k-nacci sequenceable group? That is, does a nonsimple k-nacci sequenceable group have a k-nacci sequenceable normal subgroup?

References

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