TREES FOR k-REVERSE MULTIPLES

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Let x be an n-digit, base g number

(1) $$x = \sum_{i=0}^{n-1} a_i g^i$$

with $0 \le a_i < g$ and $a_{n-1} \ne 0$. If, for some integer k, where $1 < k < g$,

(2) $$kx = \sum_{i=0}^{n-1} a_{n-1-i} g^i$$

then x is called a *k-reverse multiple*, Previously, this author showed that all k-reverse multiples may be found using rooted trees [3]. A more detailed examination of these trees is the focus of this paper.

If x is a k-reverse multiple, then we obtain from (1) and (2) the following equations

(3) $$ka_i + r_{i-1} = a_{n-1-i} + r_i g, \; i = 0, \ldots, n-1,$$

where $0 \le r_i < g$ for $i = 0, \ldots, n-2$ and $r_{-1} = r_{n-1} = 0$. Letting $i = n-1$ in (3) gives $a_0 \ne 0$ since $a_{n-1} \ne 0$. To determine whether there are any k-reverse multiples for a given g, we consider the equations in (3) two at a time. At the $(i+1)^{\text{st}}$ step, $i = 0, 1, \ldots,$ we examine the pair of equations

(4) $$\begin{cases} ka_i + r_{i-1} = a_{n-1-i} + r_i g \\ ka_{n-1-i} + r_{n-2-i} = a_i + r_{n-1-i} g \end{cases}$$

seeking nonnegative integers a_i, a_{n-1-i}, r_i, and r_{n-2-i} less than g, where r_{i-1} and r_{n-1-i} are known from the previous step. The following graphical notation is convenient. If r_{n-1-i}, r_{i-1}, a_{n-1-i}, a_i, r_{n-2-i}, and r_i satisfy (4), then we will write

(5) $$\begin{array}{l} (r_{n-1-i},\ r_{i-1}) \\ \quad\Big| \quad (a_{n-1-i},\ a_i) \\ (r_{n-2-i},\ r_i) \end{array}$$

(Implicit in this notation is the assumption that the a's and r's are nonnegative integers less than g.)

When a given g has k-reverse multiples, we are able to generate a rooted tree. We call the root of the tree $(r_{n-1}, r_{-1}) = (0, 0)$, the 0^{th} level and the node designated by (r_{n-2-i}, r_i), the $(i+1)^{\text{st}}$ level. Since $0 \le r_i < g$, there are only a finite number of possible distinct nodes. If a node is labeled with a pair that has already appeared in the tree, the tree can be pruned. The following theorem shows how a tree is used to determine k-reverse multiples. The proof appeared in [3] and hence is omitted here.

Theorem 1: For a given g, suppose there are k-reverse multiples; that is, suppose a tree exists. There is a $2i+2$-digit or a $2i+3$-digit number satisfying (2) if and only if the tree contains at the i, $i+1$, and $i+2$ levels, respectively,

$$\begin{array}{lll} (r_{n-1-i},\ r_{i-1}) & (s_{n-1-i},\ s_{i-1}) & \text{(level } i) \\ \quad \big| \ (a_{n-1-i},\ a_i) & \quad \big| \ (b_{n-1-i},\ b_i) & \\ (r,\ r) & (s,\ t) & \text{(level } i + 1) \\ & \quad \big| \ (B,\ B) & \\ & (t,\ s) & \text{(level } i + 2) \end{array}$$

where, in the second case, $B = (gs - t)/(k - 1)$. In these cases, x is given, respectively, by

$$x = a_{n-1}a_{n-2} \ \dots \ a_{n-1-i}a_i \ \dots \ a_1a_0 \qquad n = 2i + 2$$
$$x = b_{n-1}b_{n-2} \ \dots \ b_{n-1-i}Bb_i \ \dots \ b_1b_0 \qquad n = 2i + 3. \quad \square$$

Theorem 1 shows the connection between a rooted tree and k-reverse multiples. A node of the form (r, r) gives rise to a k-reverse multiple with an even number of digits. Consecutive nodes (s, t) and (t, s) produce a multiple with an odd number of digits. The following example illustrates the use of this theorem.

Example 1: $g = 10$, $k = 4$.

We begin by letting $r_{n-1} = r_{-1} = 0$ in (4) and solve the system:

$$\begin{aligned} 4a_0 \ + 0 \quad &= a_{n-1} + 10r_0 \\ 4a_{n-1} + r_{n-2} &= a_0 \quad + \ 0. \end{aligned}$$

The only solution is $r_{n-2} = 0$, $r_0 = 3$, $a_{n-1} = 2$, and $a_0 = 8$. This gives the node and edge labels for the first level of the tree. We continue in this manner and obtain the following pruned tree:

(6)
$$\begin{tikzcd} & & (0,\ 0) \arrow[d, dash, "{(2,\ 8)}"] & \\ & & (0,\ 3) \arrow[d, dash, "{(1,\ 7)}"] & \\ & & (3,\ 3) \arrow[dl, dash, "{(7,\ 1)}"'] \arrow[dr, dash, "{(9,\ 9)}"] & \\ & (3,\ 0) \arrow[d, dash, "{(8,\ 2)}"'] & & (3,\ 3) \\ & (0,\ 0) \arrow[dl, dash, "{(0,\ 0)}"'] \arrow[dr, dash, "{(2,\ 8)}"] & & \\ (0,\ 0) & & (0,\ 3) & \end{tikzcd}$$

The tree is not continued any further since (0, 0), (0, 3), and (3, 3) have appeared previously.

Observe that the node label (0, 0) follows (0, 0) at level 5, but not at level 1. This will always be the case since the equations in (4) are satisfied by the trivial or zero solution. Although $r_0 \neq 0$, the node label (0, 0) is permissible after the first level.

By Theorem 1, the node (3, 3) at the second level gives rise to the 4-digit 4-reverse multiple 2178. Moreover, the consecutive nodes (3, 3) and (3, 3) produce the 5-digit multiple 21978. Extending this portion of the pruned tree shows that all numbers of the form 219...978 are 4-reverse multiples. Thus, there are n-digit 4-reverse multiples for all $n \geq 4$.

The relationship between the node and edge labels and verifying that a specific base g number x is, in fact, a k-reverse multiple may be demonstrated by performing base g multiplication of x by k, explicitly indicating all carries from one digit to the next. It should be noted that when some x is known to be a k-reverse multiple this computation provides an alternate way to obtain some of the node labels.

For example, 21782178 is a base 10 4-reverse multiple [corresponding to the path from the root to node (0, 0) at level 4 in (6) above]. The multiplication verifying this fact is:

$$\begin{array}{cccccccc}
{\scriptstyle 0} & {\scriptstyle 3} & {\scriptstyle 3} & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 3} & {\scriptstyle 3} & {\scriptstyle 0} \\
2 & 1 & 7 & 8 & 2 & 1 & 7 & 8 \\
 & & & & & & & \times\ 4 \\
\hline
8 & 7 & 1 & 2 & 8 & 7 & 1 & 2
\end{array}$$

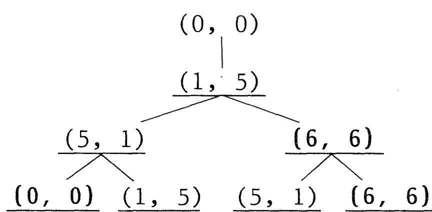
The carries from the node pairs and the digits of x form the edge labels.

From Theorem 1, the digits of 21782178 are the first elements of the edge labels from the root to node (0, 0) at level 4 followed by the second elements for the same edge labels taken from node (0, 0) at level 4 back up to the root. Similarly, the carry numbers noted above the digits of x are the elements of the node labels along the path. The first four carries are the first elements of the node labels from level 1 to level 4, and the second four carries are the second elements of the same node labels from level 3 back up to level 0. The root is always labeled (0, 0) and by Theorem 1 (since 21782178 has an even number of digits) the node label at level 4 must have both digits the same. □

The following examples illustrate some characteristics exhibited by trees for k-reverse multiples like the one shown in (6). We will use bold type for a node that determines a k-reverse multiple with an even number of digits and underlining for one that determines a k-multiple with an odd number of digits. Further, since we recognize the existence of k-reverse multiples graphically by particular types of node labels, we will omit the edge labels. There is no loss in doing this, for we may always use (4) to solve for

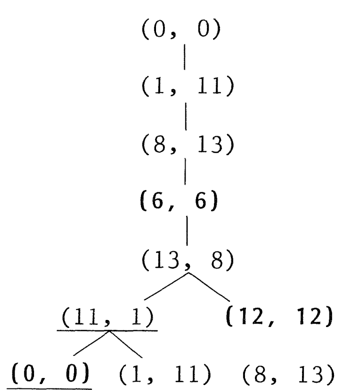
$$a_{n-1-i} = (kr_{n-1-i}g - kr_{n-2-i} + r_i g - r_{i-1})/(k^2 - 1),$$
$$a_i = (kr_i g - kr_{i-1} + r_{n-1-i}g - r_{n-2-i})/(k^2 - 1).$$

Example 2: $g = 11$, $k = 7$.



By Theorem 1, the node (1, 5) along with its child (5, 1) determines a 3-digit multiple and (6, 6), a 4-digit one. Both (5, 1), with its child (1, 5), and (6, 6), with its child (6, 6), give rise to 5-digit multiples. In fact, there are n-digit 7-reverse multiples for $n \geq 3$. □

Example 3: $g = 19$, $k = 14$.



In this case, there are n-digit 14-reverse multiples for $n = 6$ and $n \geq 10$. □

Although we require $r_i < g$, in the examples above it happens that $r_i < k$. In [3] this was shown always to be the case.

In many instances the entire pruned tree can be determined from just an initial branch. The following theorem gives one way in which this can be done.

Theorem 2: If
$$\begin{array}{c} (r,\ s) \\ \Big| \ (a,\ b) \\ (u,\ v) \end{array} \quad \text{then} \quad \begin{array}{c} (v,\ u) \\ \Big| \ (b,\ a) \\ (s,\ r) \end{array}.$$

Proof: By hypothesis, the equations in (4) must be satisfied. Switching the order of the two equations gives the desired result. □

As an illustration of Theorem 2, consider the tree in Example 3. Suppose we know
$$\begin{array}{c} (0,\ 0) \\ | \\ (1,\ 11) \\ | \\ (8,\ 13) \\ | \\ (6,\ 6) \end{array}$$

Then Theorem 2 allows us to derive
$$\begin{array}{c} (6,\ 6) \\ | \\ (13,\ 8) \\ | \\ (11,\ 1) \\ | \\ (0,\ 0) \end{array}$$

immediately without using (4).

We will use the notation
$$\begin{array}{c} [r,\ s] \\ \Big| \ [a,\ b] \\ [u,\ v] \end{array} \tag{7}$$

to indicate solely that the equations in (4) are satisfied by integers; that is,
$$\begin{cases} kb + s = a + vg, \\ ka + u = b + rg. \end{cases} \tag{8}$$

Thus, the notation in (7) does not imply that the integers are nonnegative and less than g. As before, when these latter restrictions do occur, we will use the (., .) notation instead of [., .]. The next two technical lemmas will be useful in the theorems that follow.

Lemma 1: Suppose there are integers such that
$$\begin{array}{c} [r,\ s] \\ \Big| \ [a,\ b] \\ [u,\ v] \end{array}$$

with $s,\ u < g$ and $0 < r,\ v$. Then $0 < a,\ b$.

Proof: Eliminating b from the equations in (8) and rearranging, we find
$$a(k^2 - 1) = k(rg - u) + (vg - s).$$

Thus, given the hypotheses, $0 < a$. Similarly, $0 < b$. □

Lemma 2: Suppose there are integers such that

$$\begin{array}{l} [r,\ s] \\ \Big| \ [a,\ b] \\ [u,\ v] \end{array}$$

with

$$(10) \qquad \begin{cases} 0 \le s,\ u, \\ r,\ v < k, \\ r \ne k - 1,\ v \ne k - 1,\ s \ne 0, \text{ or } u \ne 0. \end{cases}$$

Then $a,\ b < g$.

Proof: From (9) we have

$$\begin{aligned} a(k^2 - 1) &= krg - ku + vg - s \\ &\le g(kr + v) \\ &\le g(k(k - 1) + (k - 1)) \\ &= g(k^2 - 1). \end{aligned}$$

Given the restrictions in the third part of (10), one of the above two inequalities must be strict. Thus, $a < g$. Similarly, $b < g$. □

Theorem 3: If there are integers such that

$$(11) \qquad \begin{array}{l} (r,\ s) \\ \Big| \ (a,\ b) \\ (u,\ v) \end{array} \quad \text{and} \quad \begin{array}{l} (r',\ s') \\ \Big| \ (a',\ b') \\ (u',\ v') \end{array}$$

then

$$(12) \qquad \begin{array}{l} (r + r',\ s + s') \\ \qquad \Big| \ (a + a',\ b + b') \\ (u + u',\ v + v') \end{array}$$

so long as

$$\begin{cases} s + s',\ u + u' < g, \\ r + r',\ v + v' < k, \\ r + r' \ne k - 1,\ v + v' \ne k - 1,\ s + s' \ne 0, \text{ or } u + u' \ne 0. \end{cases}$$

Proof: By hypothesis, (8) must be satisfied by $r,\ s,\ \ldots$, and by $r',\ s',\ \ldots$. Adding the corresponding equations gives the desired equations for (12). Since all the numbers in (11) are nonnegative, those in (12) must be also. By Lemma 2, $a + a'$ and $b + b'$ must be less than g. □

Theorem 4: If there are integers such that

$$(13) \qquad \begin{array}{l} (r,\ s) \\ \Big| \ (a,\ b) \\ (u,\ v) \end{array} \quad \text{and} \quad \begin{array}{l} (r',\ s') \\ \Big| \ (a',\ b') \\ (u',\ v') \end{array}$$

then

$$(14) \qquad \begin{array}{l} (r - r',\ s - s') \\ \qquad \Big| \ (a - a',\ b - b') \\ (u - u',\ v - v') \end{array}$$

so long as

$$\begin{cases} 0 \le s - s',\ u - u', \\ 0 < r - r',\ v - v'. \end{cases}$$

Proof: By hypothesis, (8) must be satisfied by r, s, ..., and by r', s', ... Subtracting the corresponding equations gives the desired equations for (14). Since all the integers in (13) are less than g, those in (14) must be also. By Lemma 1, $a - a'$ and $b - b'$ are positive. □

The above theorems allow the completion of all or at least large portions of a pruned tree when only an initial piece is known. Suppose, in Example 1, only

$$\text{(15)} \qquad \begin{array}{c}(0,\ 0)\\ | \\ (0,\ 3)\\ | \\ (3,\ 3)\end{array}$$

were known. We would be able immediately to derive the rest:

$$\begin{array}{ccc} & (3,\ 3) & \\ \diagup & & \diagdown \\ (3,\ 0) & & (3,\ 3) \\ | & & \\ (0,\ 0). & & \end{array}$$

The left side follows from Theorem 2; the right from Theorem 3, since

$$\begin{array}{c}(0,\ 0)\\ | \\ (0,\ 3)\end{array} \quad \text{and} \quad \begin{array}{c}(3,\ 3)\\ | \\ (3,\ 0)\end{array} \quad \text{imply} \quad \begin{array}{c}(3,\ 3)\\ | \\ (3,\ 3).\end{array}$$

Note that by the restrictions in Lemma 2,

$$\begin{array}{c}(0,\ 0)\\ | \\ (0,\ 3)\end{array} \quad \text{and} \quad \begin{array}{c}(3,\ 0)\\ | \\ (0,\ 0)\end{array} \quad \text{do not imply} \quad \begin{array}{c}(3,\ 0)\\ | \\ (0,\ 3).\end{array}$$

Thus, we are able to derive the entire pruned tree for Example 1 knowing only (15) or, equivalently, knowing only that 2178 is a 4-reverse multiple. Similarly, a careful examination of the trees in Examples 2 and 3 shows that each follows, respectively, from the 3-digit number 118 and the 6-digit number 1 2 11 8 17 15.

It may sound very restrictive to assume that we know an initial portion of a tree. However, this is equivalent to assuming that a k-reverse multiple for a given g is known. The problem then is to find or characterize all other multiples and this is done using the associated pruned tree. Hence, if we know an n-digit k-reverse multiple for some small n, then we do know an initial portion of the tree. The problem is then to complete the tree quickly and easily. As an illustration, consider the following, more complicated, example.

Example 4: $g = 44$, $k = 27$.

The 6-digit number

(16) 1 7 18 5 24 31

is a base 44, 27-reverse multiple; this can be verified through multiplication:

$$\begin{array}{rrrrrr} {}^{4} & {}^{11} & {}^{3} & & {}^{15} & {}^{19} \\ 1 & 7 & 18 & 5 & 24 & 31 \\ & & & & & \times\ 27 \\ \hline 31 & 24 & 5 & 18 & 7 & 1 \end{array}$$

The carry numbers are numbers in the node labels of the initial portion of the tree, so we have

$$\begin{tikzcd}[row sep=small]
(0,\ 0) \arrow[d, dash] \\
(4,\ 19) \arrow[d, dash] \\
(11,\ 15) \arrow[d, dash] \\
(3,\ 3)
\end{tikzcd}$$

We complete the tree using the above theorems. For example, by Theorem 3

$$\begin{tikzcd}[row sep=small]
(4,\ 19) \arrow[d, dash] & \text{and} & (3,\ 3) \arrow[d, dash] & \text{imply} & (7,\ 22) \arrow[d, dash] \\
(11,\ 15) & & (15,\ 11) & & (26,\ 26)
\end{tikzcd}$$

In the tree that follows, the superscript j on a particular node indicates that it was derived using Theorem j, $j = 2, 3, 4$. So in the above case, we would write $(26,\ 26)^3$.

$$\begin{tikzcd}[row sep=small, column sep=tiny]
 & & & (0,\ 0) \arrow[d, dash] & & & \\
 & & & (4,\ 19) \arrow[d, dash] & & & \\
 & & & (11,\ 15) \arrow[dll, dash] \arrow[drr, dash] & & & \\
 & (3,\ 3) \arrow[d, dash] & & & & \underline{(7,\ 22)^3} \arrow[dl, dash] \arrow[dr, dash] & \\
 & (15,\ 11)^2 \arrow[d, dash] \arrow[dr, dash] & & & (22,\ 7)^4 \arrow[d, dash] & & \underline{(26,\ 26)^3} \arrow[dl, dash] \arrow[d, dash] \\
 & \underline{(19,\ 4)^2} \arrow[dl, dash] \arrow[d, dash] & (23,\ 23)^3 \arrow[d, dash] & & (15,\ 11)^2 & (22,\ 7)^2 & \underline{(26,\ 26)^3} \\
\underline{(0,\ 0)^2} & (4,\ 19)^3 & (11,\ 15)^2 & & & &
\end{tikzcd} \tag{17}$$

Note that the theorems above do not guarantee that the pruned tree of (17) is complete and that no branches are missing. The next theorem addresses this concern.

Theorem 5: Suppose g has a k-reverse multiple; further, suppose the tree contains

$$\begin{tikzcd}[row sep=small]
 & (r,\ s) \arrow[dl, dash, "{(a,\ b)}"'] \arrow[dr, dash, "{(a',\ b')}"] & \\
(u,\ v) & & (u',\ v')
\end{tikzcd} \tag{18}$$

where $u > u'$ and $v \leq v'$. Then

$$\begin{tikzcd}
{[0,\ 0]} \arrow[d, dash, "{[a - a',\ b - b']}"] \\
{[u - u',\ v - v']}
\end{tikzcd}$$

where $0 < u - u' < k$, $-k < v - v' \leq 0$, $-g < a - a' < 0$ and $-g < b - b' < 0$.

Proof: Recall that if (18) occurs in a tree, then each number in the node label must be less than k and each number in the edge label must be less than g. Thus, all the claims in the conclusion follow immediately except for $a - a'$, $b - b' < 0$. From (9) we know that

$$a - a' = (-k(u - u') + (v - v')g)/(k^2 - 1).$$

Hence, $a - a' < 0$. Similarly, $b - b' < 0$. □

Suppose, for a given g, that we know some k-reverse multiple and thus are able to obtain the initial portion of the tree. We apply Theorems 2, 3, and 4 whenever possible until all branches end with nodes that have appeared previously. At this point, we are in the position of asking if there are any missing branches. By Theorem 5, if there are *no* integers c, d, t, w for which

$$\text{(19a)}\qquad \begin{array}{l} [0,\ 0] \\ \quad\Big|\ [-c,\ -d] \\ [t,\ -w] \end{array}$$

where

$$\text{(19b)}\qquad 0 < t < k,\ 0 \le w < k,\ 0 < c < g,\ 0 < d < g,$$

then we can be assured that there are *no* missing branches in the tree.

In all the examples considered thus far, (19) is never satisfied. To verify this for Example 1, we must consider the equations

$$\text{(20)}\qquad \begin{array}{l} -4d = -c - 10w \\ -4c + t = -d. \end{array}$$

obtained from equations (4). Eliminating d in (20) gives $4t = 15c - 10w$; thus, $5 \mid t$. However, $0 < t < 5$. Consequently, there are no solutions to (20) and, hence, to (19). Thus, by Theorem 5, the tree in (6) is complete.

Theorem 6: Suppose g has a k-reverse multiple and the tree contains

$$\begin{array}{l} (r,\ s) \\ \quad\Big|\ (a,\ b) \\ (u,\ v) \end{array}$$

Further, suppose

$$\begin{array}{l} [0,\ 0] \\ \quad\Big|\ [-c,\ -d] \\ [t,\ -w] \end{array}$$

with $0 < t < k$, $0 \le w < k$, $0 < c < g$, and $0 < d < g$. Then

$$\begin{array}{l} (r,\ s) \\ \quad\Big|\ (a - c,\ b - d) \\ (u + t,\ v - w) \end{array} \qquad \text{or} \qquad \begin{array}{l} (r,\ s) \\ \quad\Big|\ (a + c,\ b + d) \\ (u - t,\ v + w) \end{array}$$

so long as either the three conditions $u + t < g$, $0 < v - w$, and $0 < r$ or the two conditions $0 < u - t$ and $v + w < k$ are fulfilled.

Proof: The first piece follows from Lemma 1; the second from Lemma 2. □

The following example illustrates the use of Theorem 6.

Example 5: $g = 40$, $k = 13$.

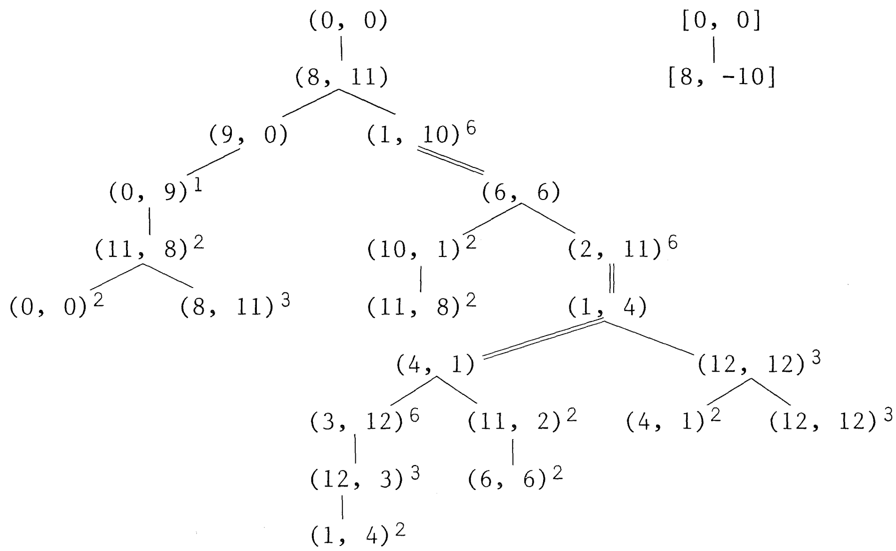
The 5-digit number

$$\text{(21)}\qquad 2\ 24\ 30\ 1\ 34$$

is a 13-reverse multiple. As in Example 4, the number in (21) gives the initial portion of the tree which has node labels (0, 0), (8, 11), and (9, 0). There is just one solution to (19); namely,

$$\begin{array}{l} [0,\ 0] \\ \quad | \\ [8,\ -10]. \end{array}$$

We now use Theorems 2, 3, 4, and 6 to complete the tree:



The double bar edges leading to nodes without a superscript indicate that none of the above theorems apply. In these cases the nodes were found using (4). Note that there are only 3 such instances. On the other hand, the 16 superscripted nodes were found easily using the theorems indicated by the superscript as in Example 4.

As we have noted, there is just one solution to (19). We used this solution in conjunction with Theorem 6 to find 3 nodes. If the tree contained any missing nodes, then by Theorem 5 equations (19) would have another solution. Since that is not the case, the tree is complete.

Acknowledgment

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