ON A THEOREM OF MONZINGO CHARACTERIZING THE PRIME DIVISORS OF CERTAIN SEQUENCES OF INTEGERS

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In [1], M. G. Monzingo extended a problem found in *Elementary Number Theory* by David M. Burton concerning the common divisors of two successive integers of the form $n^2 + 3$ by establishing

Theorem 1 (Monzingo): Let p be an odd prime. If p is of the form $4K + 1$, then p is the only prime that divides successive integers of the form $n^2 + K$, and p divides successive pairs precisely when n is of the form $bp + 2K$, for any integer b. If p is of the form $4K + 3$, then p is the largest prime that divides successive integers of the form $n^2 + (3K + 2)$, and p divides successive pairs precisely when n is of the form $bp + (2K + 1)$, for any integer b. Furthermore, p will be the only prime divisor if and only if $p = 3$.

The purpose of this note is to generalize these results to the general quadratic. Specifically, we prove the following

Theorem 2: Let p be an odd prime and define $P(n) \equiv a_2n^2 + a_1n + a_0$, where n and all coefficients are integers and $a_2 \neq 0$. If p divides $P(n)$ and $P(n + d)$, where d is an integer not divisible by p, then p divides $a_2^2d^2 - a_1^2 + 4a_0a_2$, and n satisfies the equation

$$(2n + d)a_2 + a_1 \equiv 0 \bmod p.$$

Proof: Suppose that p divides $P(n)$ and $P(n+d)$. Since p divides the difference of these integers, and p does not divide d, p divides $Q(n) \equiv (2n + d)a_2 + a_1$, i.e., n satisfies

$$(2n + d)a_2 + a_1 \equiv 0 \bmod p.$$

In addition, p divides $nQ(n) - 2P(n)$, i.e., p divides $R(n) \equiv n(a_2d - a_1) - 2a_0$. Finally, p divides $(a_2d - a_1)Q(n) - 2a_2R(n)$, and the result is established unless (perhaps) either $n = 0$ or $a_2d - a_1 = 0$. Since it is straightforward to verify directly that p divides $a_2^2d^2 - a_1^2 + 4a_0a_2$ in each of these cases, the theorem is established.

Remark: Theorem 1 (Monzingo) follows easily from Theorem 2 after selecting $d = 1$, $a_2 = 1$, $a_1 = 0$, and $a_0 = K$ or $a_0 = 3K + 2$, depending on whether $p = 4K + 1$ or $p = 4K + 3$, respectively.

Reference

1. M. G. Monzingo. "On Prime Divisors of Sequences of Integers Involving Squares." *Fibonacci Quarterly* *26.1* (1988):31-32.

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