THE GOLDEN-FIBONACCI EQUIVALENCE

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1. Introduction

The main purpose of this paper is to show the equivalence between the Golden Number and the Fibonacci Number line-sequential vector spaces. The conventions are the same as those adopted in our previous paper [6].

2. The Golden Number Line-Sequences

We shall consider the following special irrational number line-sequences:

(2.1) $F_{1,A} = F_{1,0} + AF_{0,1}$,

(2.2) $F_{1,B} = F_{1,0} + BF_{0,1}$,

where (see [6], (4.5) and (4.6):

(2.3) $A = (1 + 5^{1/2})/2$,

(2.4) $B = (1 - 5^{1/2})/2$,

(2.5) $AB = -1,\ A + B = 1$.

We shall refer to A and B as the large and the small Golden Ratios, respectively, and shall in general simply refer to these and their powers collectively as Golden Numbers.

Likewise, the ratio between the neighboring Fibonacci Numbers u_{n+1}/u_n will be called the large Fibonacci Ratio. Here, "large" means that the suffices $n + 1 > n$, without inference to the values of the u's or their ratio. Its negative reciprocal will be referred to as the small Fibonacci ratio.

The line-sequences (2.1) and (2.2) are found to be:

(2.6) $F_{1,A}$: $\ldots, A^{-3}, A^{-2}, A^{-1}, 1, A^1, A^2, A^3, \ldots$;

(2.7) $F_{1,B}$: $\ldots, B^{-3}, B^{-2}, B^{-1}, 1, B^1, B^2, B^3, \ldots$.

These are none other than a pair of divergent and convergent geometrical progressions of Golden Numbers. Henceforth, we shall refer to these two line-sequences simply as the Golden Pair. Correspondingly, the pair $F_{1,0}$ and $F_{0,1}$ will be referred to as the Fibonacci Pair.

A number of mathematical curios now begin to reveal their origin in this light.

a. On inspection of the line-sequences (2.6) and (2.7), it is obvious that Binet's formula can be obtained independently from the Golden Pair without following through the conventional algebraic derivation [7]. This is done in (4.9) below. Furthermore, as is well known, the large Fibonacci Ratio approaches the large Golden Ratio as a limit (see [7], p. 53); that is,

(2.8) $\lim u_{n+1}/u_n = A$.

b. For an arbitrary line-sequence, it has been suggested (see [2] and [3]) that the same limit as (2.8) also exists between a neighboring pair, and examples are given for $F_{1,4}$ and $F_{2,1}$. This obviously cannot be true in general, as is evidenced by the counterexample of (2.7).

3. The Golden-Fibonacci Space

By (2.8) of [6] and (2.5) above, it is obvious that both terms in the Golden Pair $F_{1,A}$ and $F_{1,B}$ are orthogonal. Hence, they form a pair of basis vectors which, like the Fibonacci Pair $F_{1,0}$ and $F_{0,1}$, spans the same 2-dimensional line-sequential vector space. Vectorally, therefore, the Golden Pair and the Fibonacci Pair are equivalent. Any line-sequence in this vector space can be expressed in terms of either of these two sets of basis vectors. In particular, the Lucas line-sequence can be expressed simply as

(3.1) $$F_{2,1} = F_{1,A} + F_{1,B}.$$

4. Some Basic Properties of the Golden Pair

Now we shall investigate some of the basic properties of the Golden Pair.

a. We note that the Golden Pair are not unit vectors; hence, they are an orthogonal but not an orthonormal pair. Their lengths are, respectively,

(4.1) $$L_{1,A} = (2 + A)^{1/2} = 1.90211\ldots,$$

(4.2) $$L_{1,B} = (2 + B)^{1/2} = 1.17557\ldots,$$

which are again in the ratio of

(4.3) $$L_{1,A}/L_{1,B} = A.$$

b. We shall investigate the linear properties of the Golden Pair. We define the following column vectors and 2×2 matrices:

(4.4) $$F = \begin{bmatrix} F_{1,0} \\ F_{0,1} \end{bmatrix}, \qquad G = \begin{bmatrix} F_{1,B} \\ F_{1,A} \end{bmatrix},$$

(4.5) $$M = \begin{bmatrix} 1 & B \\ 1 & A \end{bmatrix}, \qquad M^{-1} = (A - B)^{-1}\begin{bmatrix} A & -B \\ -1 & 1 \end{bmatrix};$$

where

(4.6) $$M^{-1}M = MM^{-1} = I,$$

(4.7) $$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, we have

(4.8) $$MF = G,$$

(4.9) $$M^{-1}G = F,$$

where the second element of (4.9) is just Binet's formula, as we have mentioned in Section 2a; and the transformation M is no more than a rotation followed by a dilation, or vice versa.

Also we have, for the lengths of the Golden Pair, the following linear transformation:

(4.10) $$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} L_{1,A}^2 \\ L_{1,B}^2 \end{bmatrix}.$$

c. The geometrical interpretation of the foregoing results is simple. Let $F_{1,0}$ and $F_{0,1}$ be the two unit vectors along the x- and y-axes, respectively. Then the Golden Pair $F_{1,A}$ and $F_{1,B}$ and the Lucas vector $F_{2,1}$ can be easily constructed as shown in Figure 1.

It is seen from the diagram that the angle of rotation from $F_{1,0}$ to $F_{1,B}$ is simply

(4.11) $$\tan^{-1} B = -31.72°.$$

Also see p. 164 of [4] or (1.6) of [1].

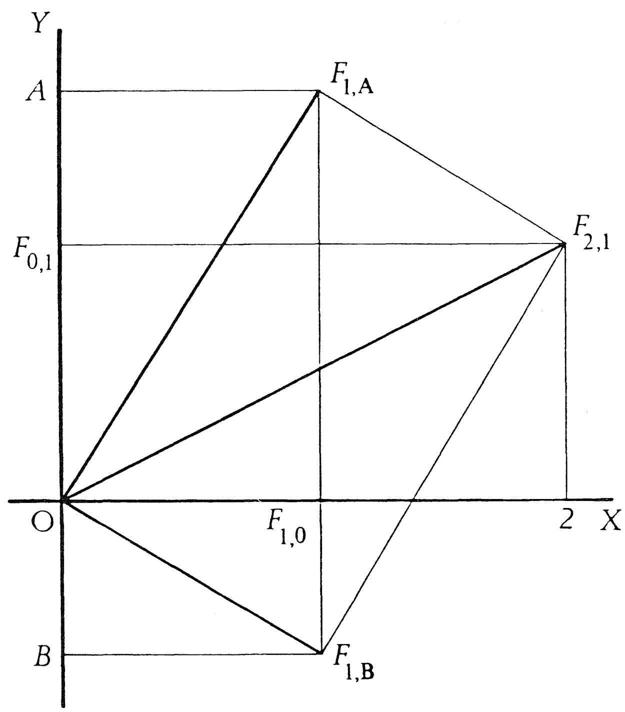

Figure 1

Geometrical Interpretation of the Fibonacci Pair and the Golden Pair

Furthermore, it is clear from (2.8) that the direction of $F_{1,A}$ is that of an asymptote toward which the Fibonacci vectors approach hyperbolically as the limit; while the vector $F_{1,B}$ lies in the direction of the other asymptote, perpendicular to the former, and alternately toward both directions of which the Fibonacci vectors recede as the limits. The Fibonacci vectors approach their limits in three different directions (see Figure 2).

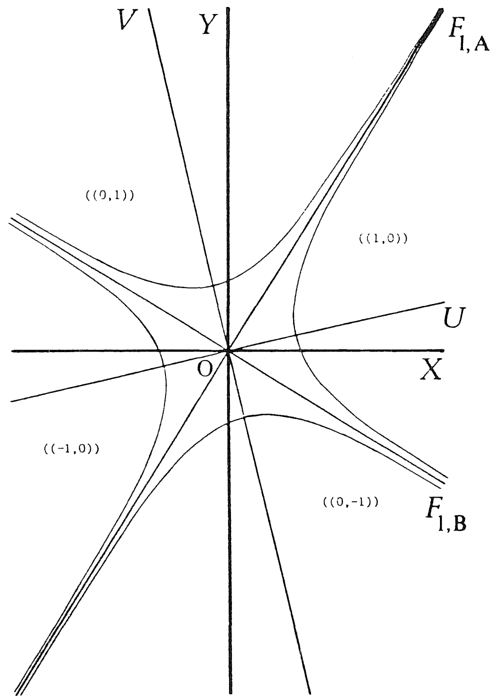

Figure 2

U, *V* are the symmetry axes; $F_{1,A}$ and $F_{1,B}$ are the asymptotes.
(Drawing is not to scale.)

Up to now, the investigation on the properties of the Fibonacci hyperbolas has been based on the ray-sequence instead of the line-sequence; thus, information about the negative branch of the line-sequence has been left out [5]. When the same procedure is applied to the entire line-sequence, a complete picture emerges. For instance, on a branch of the hyperbola

(4.12) $x^2 + xy - y^2 - 1 = 0$

lie the following set of Fibonacci points:

(4.13) ((1, 0)):

$\ldots, (5, -3), (2, -1), (1, 0), (1, 1), (2, 3), (5, 8) \ldots;$

and on a branch of the complementary hyperbola

(4.14) $x^2 + xy - y^2 + 1 = 0$

lie the complementary set of Fibonacci points

(4.15) ((0, 1)):

$\ldots, (-8, 5), (-3, 2), (-1, 1), (0, 1), (1, 2), (3, 5) \ldots .$

The two sets ((1, 0)) and ((0, 1)) make up all the neighboring pairs in the line-sequence. The remaining two branches are occupied by the sets ((-1, 0)) and ((0, -1)) of the negative Fibonacci line-sequence, as shown in Figure 2. This analysis also reveals that the parity axes (see [6], Fig. 1) correspond to the symmetry axes *U* and *V*, rather than *X* and *Y*.

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Author and Title Index for
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Currently, Dr. Charles K. Cook of the University of South Carolina at Sumter is working on an AUTHOR index, TITLE index and PROBLEM index for *The Fibonacci Quarterly*. In fact, the three indices are already completed. We hope to publish these indices in 1993 which is the 30th anniversary of *The Fibonacci Quarterly*. Dr. Cook and I feel that it would be very helpful if the publication of the indices also had AMS classification numbers for all articles published in *The Fibonacci Quarterly*. We would deeply appreciate it if all authors of articles published in *The Fibonacci Quarterly* would take a few minutes of their time and send a list of articles with primary and secondary classification numbers to

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