ARMSTRONG NUMBERS: $153 = 1^3 + 5^3 + 3^3$

Gordon L. Miller and Mary T. Whalen

University of Wisconsin, Stevens Point, WI 54481

(Submitted October 1990)

A number N is an *Armstrong number of order* n (n being the number of digits) if

$$abcd\ldots = a^n + b^n + c^n + d^n + \cdots = N.$$

The number 153 is an Armstrong number of order 3 because

$$1^3 + 5^3 + 3^3 = 1 + 125 + 27 = 153.$$

Likewise, 54748 is an Armstrong number of order 5 because

$$5^5 + 4^5 + 7^5 + 4^5 + 8^5 = 3125 + 1024 + 16807 + 1024 + 32768 = 54748.$$

More generally, an n-digit number in base b is said to be a *base* b *Armstrong number of order* n if it equals the sum of the n^{th} powers of its base b digits. In all bases, we disregard the trivial cases where $n = 1$.

A literature search revealed very little about Armstrong numbers. This set of numbers is occasionally mentioned in the literature as a number-theoretic problem for computer solution (see Spencer [1]). Only third- and fourth-order Armstrong numbers in base ten were noted. The library search did not disclose the identity of Armstrong or any circumstances relating to the discovery of this special set of numbers. Some authors have used the term *Perfect Digital Invariant* to describe these same numbers.

We wrote a computer program to find all decimal Armstrong numbers of orders 1 through 9 simply by testing each integer for the desired property. Table 1 lists the results. It is interesting to note that there are no decimal second-order Armstrong numbers. For orders 3 through 9, there are either three or four Armstrong numbers with one exception: 548,834 is the only Armstrong number of order 6.

Table 1

Decimal Armstrong Numbers Less than One Billion

153	1741725
370	4210818
371	9800817
407	9926315
1634	24678050
8208	24678051
9474	88593477
54748	146511208
92727	472335975
93084	534494836
548834	912985153

Is the set of Armstrong numbers in any base infinite? Consider the number N of n digits in base b. Then $N \geq b^{n-1}$. The Armstrong sum, AS, $\leq n(b-1)^n$.

Since

$$\frac{N}{AS} \geq \frac{b^{n-1}}{n(b-1)^n} = \frac{1}{bn}\left(\frac{b}{b-1}\right)^n,$$

and this tends to infinity as n increases, $N > AS$ for all sufficiently large values of n. Therefore, there are only finitely many Armstrong numbers in any base.

As an example, suppose we have a three-digit number in base two; that is, b is 2 and n is 3. Then $N \geq 2^2 = 4$ and $AS \leq 3(1)^2 = 3$. Therefore, it is not possible to have an Armstrong number in base two with three or more digits. The highest numbers that need to be tested to be sure of having all base-two Armstrong numbers would be the two-digit numbers. Excluding the trivial case of one-digit numbers, the only numbers that need to be tested are 10_{two} and 11_{two}, neither of which is an Armstrong number, since $1^2 + 0^2$ does not equal 10_{two} and $1^2 + 1^2$ does not equal 11_{two}. Therefore, there are no Armstrong numbers in base two.

Similarly, in base three: If $n = 8$, we have $N \geq 3^7 = 2187$ and $AS \leq 8(2)^8 = 2048$. Therefore, it is impossible to have a base-three Armstrong number of more than seven digits. One need only check the base-three numbers up to and including those of seven digits to be assured of having all Armstrong numbers in base three. It can be shown, in like manner, that $n = 13$ is sufficient to obtain all the Armstrong numbers in base four. Table 2 lists all the Armstrong numbers in bases three and four.

Table 2

All Armstrong Numbers in Base Three and Base Four

Base	3	4
	12	130
	22	131
	122	203
		223
		313
		332
		1103
		3303

The maximum number of digits that must be checked in any base grows rapidly as the base increases, and it becomes cumbersome to test all integers up to the theoretical maximum for the Armstrong property. Table 3 lists the bases from two to twenty and the maximum number of digits of the integers that would need to be checked to find all Armstrong numbers in that base.

Table 3

Maximum Number of Digits that Must Be Checked in Each Base
To Obtain All Armstrong Numbers

Base	Maximum Digits	Base	Maximum Digits
2	2	12	78
3	7	13	87
4	13	14	97
5	20	15	106
6	28	16	116
7	35	17	126
8	43	18	136
9	52	19	146
10	60	20	156
11	69		

Our computer program for bases five through nine searched all the numbers from one to one billion. The results are reported in Table 4. Note that in base five there are no Armstrong numbers with 5, 7, 8, 10, 11, or 12 digits. There is a 13-digit Armstrong number and the computer search was terminated before 14-digit base-five numerals since such a number is greater than one billion.

Table 4

Armstrong Numbers in Bases 5 through 9, Less than One Billion

Base	5	6	7	8	9
	23	243	13	24	45
	33	514	34	64	55
	103	14340	44	134	150
	433	14341	63	205	151
	2124	14432	250	463	570
	2403	23520	251	660	571
	3134	23521	305	661	2446
	124030	44405	505	40663	12036
	124031	435152	12205	42710	12336
	242423	5435254	12252	42711	14462
	434434444	12222215	13350	60007	2225764
	1143204434402	555435035	13351	62047	6275850
			15124	636703	6275851
			36034	3352072	12742452
			205145	3352272	356614800
			1424553	3451473	356614801
			1433554	4217603	1033366170
			3126542	7755336	1033366171
			4355653	16450603	1455770342
			6515652	63717005	
			125543055	233173324	
			161340144	3115653067	
			254603255	4577203604	
			336133614		
			542662326		
			565264226		
			13210651042		
			13213642035		
			13261421245		
			23662020022		

Other interesting observations from Table 4: In the range tested, there are more base-seven Armstrong numbers than there are for any other base; there are more base-eight Armstrong numbers than there are base-ten Armstrong numbers; in bases six, seven, and eight, there are no four-digit Armstrong numbers.

Armstrong numbers provide intriguing mathematical recreation. Elementary students could be asked to find Armstrong numbers in base two, base eight, or any other nondecimal base. This activity would provide practice in the operations of addition and multiplication in these bases, and lead to a better understanding of nondecimal numbers. High school students could be challenged to write computer programs which would output Armstrong numbers in any base. This latter activity affords an excellent opportunity to discuss program efficiency, since students will likely find that their programs, though logically correct, will not go beyond numbers with only a few digits before becoming overwhelmed by time-consuming calculations. The rate at which the computing time grows as a function of the number of digits, which is an important characteristic of a computer algorithm, can be introduced here.

Reference

1. Donald Spencer. *Challenging Mathematical Problems with PASCAL Solutions*. Ormand Beach, Florida: Camelot Publishing Co., 1988.

*****