ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

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Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

Dedication. This year's column is dedicated to **Dr. A. P. Hillman** in recognition of his 27 years of devoted service as editor of the Elementary Problems Section.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-724 *Proposed by Larry Taylor, Rego Park, NY*

Dedicated to Dr. A. P. Hillman

Let n be a positive integer. Prove that the numbers $L_{n-1}L_{n+1}$, $5F_n^2$, L_{3n}/L_n, L_{2n}, F_{3n}/F_n, L_n^2, $5F_{n-1}F_{n+1}$ are in arithmetic progression and find the common difference.

B-725 *Proposed by Russell Jay Hendel, Patchogue, NY and Herta T. Freitag, Roanoke, VA*

Dedicated to Dr. A. P. Hillman

(a) Find an infinite set of right triangles each of which has a hypotenuse whose length is a Fibonacci number and an area that is the product of four Fibonacci numbers.

(b) Find an infinite set of right triangles each of which has a hypotenuse whose length is the product of two Fibonacci numbers and an area that is the product of four Lucas numbers.

B-726 *Proposed by Florentin Smarandache, Phoenix, AZ*

Dedicated to Dr. A. P. Hillman

Let $d_n = P_{n+1} - P_n$, $n = 1, 2, 3, \ldots$, where P_n is the n^{th} prime. Does the series

$$\sum_{n=1}^{\infty} \frac{1}{d_n}$$

converge?

B-727 *Proposed by Ioan Sadoveanu, Ellensburg, WA*

Dedicated to Dr. A. P. Hillman

Find the general term of the sequence (a_n) defined by the recurrence

$$a_{n+2} = \frac{a_{n+1} + a_n}{1 + a_{n+1}a_n}$$

with initial values $a_0 = 0$ and $a_1 = (e^2 - 1)/(e^2 + 1)$, where e is the base of natural logarithms.

B-728 *Proposed by Leonard A. G. Dresel, Reading, England*

Dedicated to Dr. A. P. Hillman

If $p > 5$ is a prime and n is an even integer, prove that

(a) if $L_n \equiv 2 \pmod{p}$, then $L_n \equiv 2 \pmod{p^2}$;
(b) if $L_n \equiv -2 \pmod{p}$, then $L_n \equiv -2 \pmod{p^2}$.

B-729 *Proposed by Lawrence Somer, Catholic University of America, Washington, D.C.*

Dedicated to Dr. A. P. Hillman

Let (H_n) denote the second-order recurrence defined by

$$H_{n+2} = aH_{n+1} + bH_n,$$

where $H_0 = 0$, $H_1 = 1$, and a and b are integers. Let p be a prime such that $p \nmid b$. Let k be the least positive integer such that $H_k \equiv 0 \pmod{p}$. (It is well-known that k exists.) If $H_n \not\equiv 0 \pmod{p}$, let $R_n \equiv H_{n+1}H_n^{-1} \pmod{p}$.

(a) Show that $R_n + R_{k-n} \equiv a \pmod{p}$ for $1 \le n \le k - 1$.
(b) Show that $R_nR_{k-n-1} \equiv -b \pmod{p}$ for $1 \le n \le k - 2$.

Acknowledgment

The editor of Elementary Problems and Solutions wishes to thank Clark Kimberling for his help in proofreading material for this section.

SOLUTIONS

A Radical Limit

B-698 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

Consider the sequence of real numbers $a_1, a_2, \ldots$, where $a_1 > 2$ and

(1) $$a_{n+1} = a_n^2 - 2 \quad \text{for } n \ge 1.$$

Find $\lim_{n\to\infty} b_n$, where

(2) $$b_n = \frac{a_{n+1}}{a_1a_2\ldots a_n} \quad \text{for } n \ge 1.$$

Solution 1 by Hans Kappus, Rodersdorf, Switzerland

We claim that

$$\lim_{n\to\infty} b_n = \sqrt{a_1^2 - 4}.$$

This follows from the formula

$$b_n^2 = \frac{(a_1^2 - 4)a_{n+1}^2}{a_{n+1}^2 - 4} \tag{3}$$

and the obvious fact that $\{a_n\}$ is an increasing sequence so $\lim\limits_{n\to\infty} a_n = \infty$.

To prove (3), we proceed by mathematical induction. We have

$$b_1^2 = \frac{a_2^2}{a_1^2} = \frac{(a_1^2 - 4)a_2^2}{(a_1^2 - 4)a_1^2} = \frac{(a_1^2 - 4)a_2^2}{(a_1^2 - 2)^2 - 4} = \frac{(a_1^2 - 4)a_2^2}{a_2^2 - 4}$$

so formula (3) is true for $n = 1$. Assume now that (3) holds for some integer $n = k - 1$. Then, from $b_k = b_{k-1}a_{k+1}/a_k^2$, we have

$$b_k^2 = \frac{b_{k-1}^2 a_{k+1}^2}{a_k^4} = \frac{(a_1^2 - 4)a_{k+1}^2}{(a_k^2 - 4)a_k} = \frac{(a_1^2 - 4)a_{k+1}^2}{(a_k^2 - 2)^2 - 4} = \frac{(a_1^2 - 4)a_{k+1}^2}{a_{k+1}^2 - 4},$$

which completes the induction.

Solution 2 by Ioan Sadoveanu, Ellensburg, WA

Using the recurrence relation in the form

$$a_{n+1} - a_n = (a_n + 1)(a_n - 2)$$

implies, by induction, that $a_{n+1} > a_n > 2$ for all $n \geq 1$.

Let x_n be defined by

$$a_n/2 = \cosh x_n. \tag{4}$$

This is possible since the hyperbolic cosine defined on $(0, \infty)$ and valued in $(1, \infty)$ is a one-to-one function.

We recall some facts concerning hyperbolic functions [1]:

$$\cosh z = \frac{e^z + e^{-z}}{2} \tag{5}$$

$$\sinh z = \frac{e^z - e^{-z}}{2} \tag{6}$$

$$\cosh^2 z - \sinh^2 z = 1 \tag{7}$$

$$\sinh 2z = 2 \sinh z \cosh z \tag{8}$$

$$\cosh 2z = 2 \cosh^2 z - 1 \tag{9}$$

$$\coth z = \frac{\cosh z}{\sinh z} \tag{10}$$

Applying (4) to (1) gives

$$2 \cosh x_n = (2 \cosh x_{n-1})^2 - 2$$

or

$$\cosh x_n = 2 \cosh^2 x_{n-1} - 1 = \cosh 2x_{n-1}$$

by (7). Thus, $x_n = 2x_{n-1}$. Repeated application of this formula yields

$$x_n = 2^{n-1}x_1.$$

Now

$$a_1 a_2 \cdots a_n = (2 \cosh x_1)(2 \cosh 2x_1) \cdots (2 \cosh 2^{n-1}x_1)$$

$$= \frac{\sinh 2x_1}{\sinh x_1} \frac{\sinh 4x_1}{\sinh 2x_1} \cdots \frac{\sinh 2^n x_1}{\sinh 2^{n-1}x_1} = \frac{\sinh 2^n x_1}{\sinh x_1}$$

using (6) and cancelling. Therefore,

$$b_n = \frac{a_{n+1}}{a_1 a_2 \cdots a_n} = \frac{\sinh x_1 (2 \cosh x_{n+1})}{\sinh 2^n x_1}$$

$$= \frac{\sinh x_1}{\sinh x_{n+1}}(2 \cosh x_{n+1}) = 2 \sinh x_1 \coth x_{n+1}.$$

But

$$\lim_{x \to \infty} \coth x = \lim_{x \to \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x \to \infty} \frac{1 + \dfrac{1}{e^{2x}}}{1 - \dfrac{1}{e^{2x}}} = \frac{1 + 0}{1 - 0} = 1.$$

Thus,

$$\lim_{n \to \infty} b_n = 2 \sinh x_1 \lim_{n \to \infty} \coth x_{n+1} = 2 \sinh x_1$$

$$= 2\sqrt{\cosh^2 x_1 - 1} = \sqrt{4 \cosh^2 x_1 - 4} = \sqrt{a_1^2 - 4}.$$

Reference

1. Abramowitz & Stegun. *Handbook of Mathematical Functions*. National Bureau of Standards, Washington, DC, 1966.

Also solved by Paul S. Bruckman, Blagoj S. Popov, and the proposer.

A Solution Using Periodic Orbits

B-699 *Proposed by Larry Blaine, Plymouth State College, Plymouth, NH*

Let a be an integer greater than 1. Define a function $p(n)$ by

$$p(1) = a - 1 \quad \text{and} \quad p(n) = a^n - 1 - \sum p(d) \quad \text{for } n \geq 2,$$

where $\sum$ denotes the sum over all d with $1 \leq d < n$ and $d|n$.
Prove or disprove that $n|p(n)$ for all positive integers n.

Solution by the proposer

Consider the function $f:[0, 1) \to [0, 1)$ defined by

$$f(x) \equiv ax \pmod{1},$$

i.e.,

$$f(x) = ax - k \text{ for } k/a \leq x < (k + 1)/a,\ k = 0, 1, \ldots, a - 1.$$

We use the customary notation

$$f^1(x) = f(x), \quad f^{n+1}(x) = f(f^n(x)) \quad \text{for } n = 1, 2, \ldots,$$

and for $x \in [0, 1)$ we define the orbit of x to be the sequence

$$x_0 = x, \quad x_n = f^n(x) \quad \text{for } n = 1, 2, \ldots .$$

We say that x is an n-periodic point if $x_0 = x_n$, but $x_0 \neq x_i$ for $i = 1, 2, \ldots, n - 1$.

Now, if x is n-periodic, then $f^n(x) = x$. The converse is not quite true: $f^n(x) = x$ if and only if x is d-periodic for some positive integer d for which $d|n$ (including, of course, $d = 1$ and $d = n$). An easy calculation shows that there are exactly $a - 1$ 1-periodic points and $a^n - 1$ points for which $f^n(x) = x$. It follows by induction that $p(n)$ is the number of n-periodic points. Since these points fall into equivalence classes (periodic orbits) of the form $\{x_0, x_1, \ldots, x_{n-1}\}$, it follows that $n|p(n)$ in all cases.

Also solved by Paul S. Bruckman and Russell Jay Hendel.

The proposer asks whether a proof can be given using elementary number theoretic techniques. Although our two other solvers gave proofs using "elementary" number theory, their proofs were not as simple as the proposer's. Hendel's proof ran for three pages and Bruckman's proof involved the Möbius inversion formula and a generalized form of Fermat's Little Theorem.

But It Doesn't Look Symmetric

B-700 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that for positive integers m and n,

$$\alpha^m(\alpha L_n + L_{n-1}) = \alpha^n(\alpha L_m + L_{m-1}).$$

Solution by Paul S. Bruckman, Edmonds, WA

Let $f(m, n) = \alpha^m(\alpha L_n + L_{n-1})$. Using $\alpha\beta = -1$, we find that

$$\begin{aligned}\alpha L_n + L_{n-1} &= \alpha(\alpha^n + \beta^n) + \alpha^{n-1} + \beta^{n-1}\\ &= \alpha^{n+1} - \beta^{n-1} + \alpha^{n-1} + \beta^{n-1}\\ &= \alpha^n(\alpha - \beta) = \alpha^n\sqrt{5}.\end{aligned}$$

Therefore, $f(m, n) = \alpha^{m+n}\sqrt{5}$, from which we see that $f(m, n) = f(n, m)$.

Solvers found various methods of showing that $f(m, n)$ is symmetric:
Melham showed that $f(m, n) = \alpha^{m+n+1} + \alpha^{m+n-1}$.
Singh showed that $f(m, n) = \alpha^{m+n-1}(\alpha^2 + 1)$.
Brown notes that the result follows from problem B-538 ($\sqrt{5}\alpha^n = \alpha L_n + L_{n-1}$).
Haukkanen generalized by showing that the following are symmetric in m and n:

$$\beta^m(\beta L_n + L_{n-1}), \quad \alpha^m(\alpha F_n + F_{n-1}), \quad \beta^m(\beta F_n + F_{n-1}).$$

Also solved by Michel Ballieu, Brian D. Beasley, Glenn Bookhout, Scott H. Brown, Russell Euler, C. Georghiou, Pentti Haukkanen, Russell Jay Hendel, Joseph J. Kostal, Graham Lord, Ray Melham, Blagoj S. Popov, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

A Pair of Triangles with Common Sides

B-701 *Proposed by Herta T. Freitag, Roanoke, VA*

In triangles ABC and DEF, $AC = DF = 5F_{2n}$, $BC = L_{n+2}L_{n-1}$, $EF = L_{n+1}L_{n-2}$, and $AB = DE = 5F_{2n+1} - L_{2n+1} + (-1)^{n-1}$. Prove that $\angle ACB = \angle DFE$.

Solution by the proposer

Let $L_{n+1} = x$, $L_n = y$, $b = 5F_{2n}$, $c = AB$, $a = L_{n+2}L_{n-1}$, $d = L_{n+1}L_{n-2}$, $e = DF$, and $f = DE$. Since

$$L_{n+2}L_{n-1} = (L_{n+1} + L_n)(L_{n+1} - L_n),$$
$$5F_{2n} = 5F_nL_n = (L_{n+1} + L_{n-1})L_n = (2L_{n+1} - L_n)L_n,$$
$$L_{n+1}L_{n-2} = L_{n+1}(2L_n - L_{n+1}),$$

and

$$5F_{2n+1} - L_{2n+1} + (-1)^{n-1} = L_{2n} + L_{2n+2} - L_{2n+1} + (-1)^{n-1},$$

and where, furthermore,

$$L_{2n} + L_{2n+2} = L_{n+1}^2 + L_n^2 \quad \text{and} \quad L_{2n+1} + (-1)^n = L_{n+1}L_n,$$

we therefore have

$$a = x^2 - y^2, \quad b = y(2x - y) = e, \quad c = x^2 - xy + y^2 = f, \quad d = x(2y - x).$$

Now, using the Law of Cosines for triangle ABC, we find

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

However, $a^2 + b^2 - c^2 = 2x^3y - x^2y^2 - 2xy^3 + y^4 = ab$. Thus, $\cos C = 1/2$; hence $\angle C = \pi/3$.

Similarly, for triangle DEF,

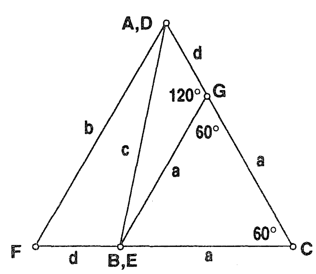
$$d^2 + e^2 - f^2 = -2x^3y + 5x^2y^2 - 2xy^3 = de,$$

from which we get $\cos F = 1/2$; hence $\angle F = \angle C = \pi/3$.

Comment by the editor:

With the same notation as in Solution 1, it is straightforward to show that $c^2 = a^2 + ad + d^2$ and $a + d = b$.

By the Law of Cosines, this tells us that there is a triangle ABG with sides of length $AB = c$, $BG = a$, and $GA = d$ and that $\angle AGB = 120°$.



Extend side AG past G for a distance a to the point C. Then, since $\angle BGC = 60°$, triangle BGC is equilateral and $BC = a$. Draw a line through A parallel to BG. and meeting CB extended at F. Thus, $\angle AFC = \angle GBC = 60°$; therefore, $\triangle AFC$ is also equilateral. Thus $BF = d$ and $AF = a + d = b$.

Giving labels D and E to points A and B, respectively, we thus see our two triangles ABC and DEF of the problem proposal and have also shown that $\angle ACB = \angle DFE = 60°$.

Also solved by Paul S. Bruckman, C. Georghiou, Russell Jay Hendel, Ray Melham, Bob Prielipp, and H.-J. Seiffert. A wonderful 6-page solution (with 7 lemmas) was also received, but the solver forgot to print his or her name on the solution sheets, so proper credit cannot be assigned.

A Comparison of Continued Fractions

B-702 *Proposed by L. Kuipers, Sierre, Switzerland*

For n a positive integer, let

$$x_n = F_n + \cfrac{1}{L_n + \cfrac{1}{F_n + \cfrac{1}{L_n + \ddots}}} \quad \text{and} \quad y_n = F_n + \cfrac{1}{F_{n+1} + \cfrac{1}{F_n + \cfrac{1}{F_{n+1} + \ddots}}}.$$

(a) Find closed form expressions for x_n and y_n.
(b) Prove that $x_n < y_n$ when $n > 1$.

Solution to part (a) by C. Georghiou, University of Patras, Patras, Greece

Assuming convergence, we have

$$x_n = F_n + \cfrac{1}{L_n + \cfrac{1}{x_n}} \quad \text{and} \quad y_n = F_n + \cfrac{1}{F_{n+1} + \cfrac{1}{y_n}}.$$

Solving these equations for x_n and y_n, respectively, we find

$$x_n = \frac{F_n}{2}\left[1 + \sqrt{1 + \frac{4}{F_n L_n}}\right] \quad \text{and} \quad y_n = \frac{F_n}{2}\left[1 + \sqrt{1 + \frac{4}{F_n F_{n+1}}}\right].$$

(The negative roots of the quadratics must be rejected since x_n and y_n are clearly positive.)

Solution to part (b) by Sahib Singh, Clarion University of Pennsylvania, Clarion, PA

It is obvious that $x_1 = y_1$. For $n > 1$, from the well-known formula $L_n = F_{n+1} + F_{n-1}$, we see that $F_{n+1} < L_n$. Applying this inequality to the formulas for x_n and y_n shows that $y_n > x_n$ when $n > 1$.

Most solvers ignored the question of convergence. Unless you know that the continued fractions converge, the operations above cannot be justified.

Proof of convergence by H.-J. Seiffert, Berlin, Germany

For positive integers a and b, let

$$z = a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \ddots}}}$$

If p_k/q_k denotes the k^{th} convergent of z, then, for all positive integers k:

$$p_0 = a, \quad p_1 = ab + 1, \quad q_0 = 1, \quad q_1 = b,$$
$$p_{2k} = ap_{2k-1} + p_{2k-2}, \quad p_{2k+1} = bp_{2k} + p_{2k-1},$$
$$q_{2k} = aq_{2k-1} + q_{2k-2}, \quad q_{2k+1} = bq_{2k} + q_{2k-1}.$$

It follows that the sequences (p_{2k}) and (q_{2k}) satisfy

$$p_0 = a, \quad p_2 = a(ab + 2), \quad p_{2k} = (ab + 2)p_{2k-2} - p_{2k-4},$$
$$q_0 = 1, \quad q_2 = ab + 1, \quad q_{2k} = (ab + 2)q_{2k-2} - q_{2k-4}.$$

These are second-order linear recurrences; and using standard methods, we find that

$$p_{2k} = a(t_1^{k+1} - t_2^{k+1})/D \quad \text{and} \quad q_{2k} = ((t_1 - 1)t_1^k - (t_2 - 1)t_2^k)/D$$

where $t_1 = (ab+2+D)/2$ and $t_2 = (ab+2-D)/2$ are the roots of $t^2 - (ab+2)t + 1 = 0$ and $D = \sqrt{ab(ab + 4)}$.

Since $t_1 > t_2 > 0$, we find that

$$\lim_{k\to\infty} \frac{p_{2k}}{q_{2k}} = \lim_{k\to\infty} \frac{a(t_1^{k+1} - t_2^{k+1})}{(t_1 - 1)t_1^k - (t_2 - 1)t_2^k} = \lim_{k\to\infty} \frac{a\left(1 - \left(\frac{t_2}{t_1}\right)^{k+1}\right)}{(t_1 - 1)\frac{1}{t_1} - \frac{t_2 - 1}{t_1}\left(\frac{t_2}{t_1}\right)^k}$$
$$= \frac{a}{1 - \frac{1}{t_1}} = \frac{at_1}{t_1 - 1}.$$

In a similar manner, we find that

$$\lim_{k\to\infty} \frac{p_{2k+1}}{q_{2k+1}}$$

has this same value. Thus,

$$\lim_{k\to\infty} \frac{p_k}{q_k}$$

exists and the continued fraction converges to this value.

One could also have noted convergence by quoting from a standard text on continued fractions, such as Theorem 3.5 from [1], which states that any simple continued fraction (positive entries and 1's in the numerators) converges. Seiffert's proof, though, not only proves convergence and finds the limit, but also gives the value of all the convergents.

Reference

1. C. D. Olds. *Continued Fractions*. Washington, D.C.: Mathematical Association of America (New Mathematics Library), 1963.

Also solved by Charles Ashbacher, Paul S. Bruckman, Russell Euler, Herta T. Freitag, C. Georghiou, Russell Jay Hendel, Hans Kappus, Carl Libis, Graham Lord, Ray Melham, Bob Prielipp, H.-J. Sieffert, Sahib Singh, and the proposer.

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