ON THE (2, F) GENERALIZATIONS OF THE FIBONACCI SEQUENCE

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A generalization of the Fibonacci sequence to vectors was defined in Atanassov, Atanassova, & Sasselov [1]. In a later article, Atanassov [2] defined the four distinct (2, F) generalizations of the Fibonacci sequence and determined a solution for one of the cases in terms of the greatest integer function. Subsequently Lee & Lee [3] published solutions for all four (2, F) generalizations using the function $f(n) = t_j$, where $j = n \mod(k) + 1$ and t_j is the j^{th} element of an ordered k-tuple $[t_1, t_2, \ldots, t_k]$. The purpose of this paper is to present a solution to each of the four (2, F) generalizations of the Fibonacci sequence as

(1) A linear combination of two second-order recursive sequences, and
(2) a polynomial in α and β and sometimes ω and $\overline{\omega}$, where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, and ω and $\overline{\omega}$ are the usual complex cube roots of 1.

In order to find a solution to the four (2, F) generalizations of the Fibonacci sequence, the following lemma is used.

Lemma: Let $p(x) = 1 \mp x \mp x^2$. The four recursive sequences defined by the four possible generating functions $1/p(x)$ have the properties given in Table 1 below, where ω and $\overline{\omega}$ are the complex cube roots of unity and $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$.

Table 1

Generating Function	General Term	Generated Series	Recursion Relation
$\dfrac{1}{1 - x - x^2}$	$F_n = \dfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$	$\sum_{n=0}^{\infty} F_n x^n$	$F_{n+2} = F_{n+1} + F_n$
$\dfrac{1}{1 + x + x^2}$	$T_n = \dfrac{\omega^{n+1} - \overline{\omega}^{n+1}}{\omega - \overline{\omega}}$	$\sum_{n=0}^{\infty} T_n x^n$	$-T_{n+2} = T_{n+1} + T_n$
$\dfrac{1}{1 - x + x^2}$	$S_n = (-1)^n \dfrac{\omega^{n+1} - \overline{\omega}^{n+1}}{\omega - \overline{\omega}}$	$\sum_{n=0}^{\infty} S_n x^n$	$S_{n+2} = S_{n+1} - S_n$
$\dfrac{1}{1 + x - x^2}$	$G_n = (-1)^n \dfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$	$\sum_{n=0}^{\infty} G_n x^n$	$G_{n+2} = -G_{n+1} + G_n$

The proof of the lemma is not shown; however, the lemma can be proved by separating the generating functions into fractions with linear denominators and then applying the binomial theorem for negative exponents. Note that, in the table,

$$F_0 = 1,\ F_1 = 1, \text{ and } F_{n+2} = F_{n+1} + F_n \text{ for } n = 2, 3, 4, \ldots .$$

From the table, it is immediate that

$$G_n = (-1)^n F_n \quad \text{and} \quad S_n = (-1)^n T_n.$$

It is also true that all four sequences may be extended to negative indices.

Theorem: Let $P_n^1 = (X_n, Y_n)$ and $P_n^2 = (Y_n, X_n)$. Then the difference equation

$$P_{n+2}^1 = P_{n+1}^j + P_n^k, \; n \ge 0; \text{ for } j, k \in \{1, 2\}$$

with the initial conditions $P_0^1 = (a, c)$, $P_1^1 = (b, d)$, where a, b, c, and d are arbitrary real numbers, defines the four distinct (2, F) generalizations of the Fibonacci sequence.

Proof of the Theorem: The four distinct cases are considered separately.

Case 1: Let $j = 1$ and $k = 1$. The system is

$$X_{n+2} = X_{n+1} + X_n, \quad n \ge 0,$$
$$Y_{n+2} = Y_{n+1} + Y_n, \quad n \ge 0, \text{ with}$$
$$P_0^1 = (a, c) \text{ and } P_1^1 = (b, d).$$

Here, the system is separable into two independent difference equations with each equation defining a generalized Fibonacci sequence. The required solution is

$$X_n = aF_{n-2} + bF_{n-1} \quad \text{and} \quad Y_n = cF_{n-2} + dF_{n-1} \quad \text{for } n \ge 0.$$

Binet's formulas are

$$X_n = a\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + b\frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad Y_n = c\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + d\frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Case 2: Let $j = 1$ and $k = 2$. The system is

$$X_{n+2} = X_{n+1} + Y_n, \quad n \ge 0,$$
$$Y_{n+2} = Y_{n+1} + X_n, \quad n \ge 0, \text{ with}$$
$$P_0^1 = (a, c) \text{ and } P_1^1 = (b, d).$$

Assuming a solution of the form

$$X = f(x) = \sum_{i=0}^{\infty} X_i x^i, \quad Y = g(x) = \sum_{i=0}^{\infty} Y_i x^i,$$

and substituting into the above system yields the system

$$(1 - x)f(x) - x^2 g(x) = a + (b - a)x$$
$$-x^2 f(x) + (1 - x)g(x) = c + (d - c)x$$

defining $f(x)$ and $g(x)$. Solving this system and applying partial fractions results in the following generating functions for $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left[\frac{(a + c) + (-a - c + b + d)x}{1 - x - x^2} + \frac{(a - c) + (-a + c + b - d)x}{1 - x + x^2}\right],$$

$$g(x) = \frac{1}{2}\left[\frac{(a + c) + (-a - c + b + d)x}{1 - x - x^2} + \frac{(-a + c) + (a - c - b + d)x}{1 - x + x^2}\right].$$

Applying the lemma and collecting terms, the equations are

$$f(x) = \frac{1}{2}\sum_{i=0}^{\infty} [(a + c)F_{i-2} + (-a + c)S_{i-2} + (b + d)F_{i-1} + (b - d)S_{i-1}]x^i$$

and

$$g(x) = \frac{1}{2}\sum_{i=0}^{\infty} [(a + c)F_{i-2} + (a - c)S_{i-2} + (b + d)F_{i-1} + (-b + d)S_{i-1}]x^i.$$

Consequently,

$$X_n = \frac{1}{2}[(a + c)F_{n-2} + (-a + c)S_{n-2} + (b + d)F_{n-1} + (b - d)S_{n-1}],$$

$$Y_n = \frac{1}{2}[(a + c)F_{n-2} + (a - c)S_{n-2} + (b + d)F_{n-1} + (-b + d)S_{n-1}].$$

Substituting

$$F_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \quad \text{and} \quad S_n = (-1)^n \frac{\omega^{n+1} - \overline{\omega}^{n+1}}{\omega - \overline{\omega}}$$

from the Lemma yields the analogs of Binet's formulas:

$$X_n = \frac{1}{2}\left[(a + c)\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) + (b + d)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\right.$$
$$\left. + (-a + c)(-1)^{n-2}\left(\frac{\omega^{n-1} - \overline{\omega}^{n-1}}{\omega - \overline{\omega}}\right) + (b - d)(-1)^{n-1}\left(\frac{\omega^n - \overline{\omega}^n}{\omega - \overline{\omega}}\right)\right]$$

and

$$Y_n = \frac{1}{2}\left[(a + c)\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) + (b + d)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\right.$$
$$\left. + (a - c)(-1)^{n-2}\left(\frac{\omega^{n-1} - \overline{\omega}^{n-1}}{\omega - \overline{\omega}}\right) + (-b + d)(-1)^{n-1}\left(\frac{\omega^n - \overline{\omega}^n}{\omega - \overline{\omega}}\right)\right].$$

Case 3: For $j = 2$ and $k = 1$, the system is

$$X_{n+2} = Y_{n+1} + X_n, \quad n \geq 0,$$
$$Y_{n+2} = X_{n+1} + Y_n, \quad n \geq 0, \text{ with}$$
$$P_0^1 = (a, c) \text{ and } P_1^1 = (b, d).$$

Assuming a solution of the form

$$X = f(x) = \sum_{i=0}^{\infty} X_i x^i, \qquad Y = g(x) = \sum_{i=0}^{\infty} Y_i x^i,$$

substituting into the system, solving for $f(x)$ and $g(x)$ and then applying partial fractions gives the generating functions in the following forms:

$$f(x) = \frac{1}{2}\left[\frac{(a + c) + (-a - c + b + d)x}{1 - x - x^2} + \frac{(a - c) + (a - c + b - d)x}{1 + x - x^2}\right],$$

$$g(x) = \frac{1}{2}\left[\frac{(a + c) + (-a - c + b + d)x}{1 - x - x^2} + \frac{(-a + c) + (-a + c - b + d)x}{1 + x - x^2}\right].$$

Applying the Lemma, collecting terms, and using the recursion relations from the Lemma yields the following forms for the generating functions:

$$f(x) = \frac{1}{2}\sum_{i=0}^{\infty}[(a + c)F_{i-2} + (a - c)G_{i-2} + (b + d)F_{i-1} + (b - d)G_{i-1}]x^i,$$

$$g(x) = \frac{1}{2}\sum_{i=0}^{\infty}[(a + c)F_{i-2} + (c - a)G_{i-2} + (b + d)F_{i-1} + (d - b)G_{i-1}]x^i.$$

Consequently,

$$X_n = \frac{1}{2}[(a + c)F_{n-2} + (a - c)G_{n-2} + (b + d)F_{n-1} + (b - d)G_{n-1}]$$

and

$$Y_n = \frac{1}{2}[(a + c)F_{n-2} + (c - a)G_{n-2} + (b + d)F_{n-1} + (d - b)G_{n-1}].$$

Substituting for F_n and G_n in terms of α and β gives the following analogs of Binet's formulas:

$$X_n = \frac{1}{2}\left[(a + c)\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) + (a - c)(-1)^n\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right)\right.$$
$$\left. + (b + d)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) + (b - d)(-1)^{n-1}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\right],$$

and

$$Y_n = \frac{1}{2}\left[(a + c)\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) + (a - c)(-1)^{n-1}\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right)\right.$$
$$\left. + (b + d)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) + (b - d)(-1)^n\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)\right].$$

Note that $G_i = (-1)^i F_i$. Collecting terms in a, b, c, and d gives

$$X_n = \frac{1}{2}[aF_{n-2}[1 + (-1)^n] + cF_{n-2}[1 - (-1)^n]$$
$$+ bF_{n-1}[1 + (-1)^{n-1}] + dF_{n-1}[1 - (-1)^{n-1}]]$$

and a similar form for Y_n.

<u>Case 4</u>: For $j = 2$ and $k = 2$, the system is

$$X_{n+2} = Y_{n+1} + Y_n, \quad n \geq 0,$$
$$Y_{n+2} = X_{n+1} + X_n, \quad n \geq 0, \text{ with}$$
$$P_0^1 = (a, c) \text{ and } P_1^1 = (b, d).$$

Again, assuming a solution of the form

$$X = f(x) = \sum_{i=0}^{\infty} X_i x^i, \qquad Y = g(x) = \sum_{i=0}^{\infty} Y_i x^i,$$

substituting into the system, solving for $f(x)$ and $g(x)$, and using partial fractions gives the following forms of the generating functions:

$$f(x) = \frac{1}{2}\left[\frac{(a + c) + (-a - c + b + d)x}{1 - x - x^2} + \frac{(a - c) + (a - c + b - d)x}{1 + x + x^2}\right],$$

$$g(x) = \frac{1}{2}\left[\frac{(a + c) + (-a - c + b + d)x}{1 - x - x^2} + \frac{(-a + c) + (-a + c - b + d)x}{1 + x + x^2}\right].$$

Applying the series from the Lemma, collecting terms, and using the recursion relations from the Lemma to combine terms gives

$$f(x) = \frac{1}{2}\sum_{i=0}^{\infty}[(a + c)F_{i-2} + (-a + c)T_{i-2} + (b + d)F_{i-1} + (b - d)T_{i-1}]x^i,$$

$$g(x) = \frac{1}{2}\sum_{i=0}^{\infty}[(a + c)F_{i-2} + (a - c)T_{i-2} + (b + d)F_{i-1} + (-b + d)T_{i-1}]x^i.$$

Thus,

$$X_n = \frac{1}{2}[(a + c)F_{n-2} + (-a + c)T_{n-2} + (b + d)F_{n-1} + (b - d)T_{n-1}]$$

and

$$Y_n = \frac{1}{2}[(a + c)F_{n-2} + (a - c)T_{n-2} + (b + d)F_{n-1} + (-b + d)T_{n-1}].$$

Substituting for F_n and T_n in terms of α, β, ω, and $\overline{\omega}$ gives the analogs of Binet's formulas:

$$X_n = \frac{1}{2}\left[(a + c)\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) + (-a + c)\left(\frac{\omega^{n-1} - \overline{\omega}^{n-1}}{\omega - \overline{\omega}}\right)\right.$$
$$\left. + (b + d)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) + (b - d)\left(\frac{\omega^n - \overline{\omega}^n}{\omega - \overline{\omega}}\right)\right]$$

and

$$Y_n = \frac{1}{2}\left[(a + c)\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) + (a - c)\left(\frac{\omega^{n-1} - \overline{\omega}^{n-1}}{\omega - \overline{\omega}}\right)\right.$$
$$\left. + (b + d)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) + (-b + d)\left(\frac{\omega^n - \overline{\omega}^n}{\omega - \overline{\omega}}\right)\right].$$

In this paper we have expressed the solutions to the (2, F) generalizations of the Fibonacci sequence as a linear combination of the terms of two recursive sequences of order 2. Since the coefficients of the terms of the recursive sequences are linear functions of the initial terms of the (2, F) sequences, it is possible to rearrange the solutions into the form of a linear combination of the initial terms, where coefficients are functions of the terms of the second-order sequences involved.

References

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A Short History on Edouard Lucas

In "Pascals's Triangle and the Tower of Hanoi" by Andreas M. Hinz, The American Mathematical Monthly, Vol 99.6 (1992) pages 538-544, one can find a very short but well written history on Edouard Lucas. It is certainly worth reading.

Gerald E. Bergum, Editor