ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-730 ***Proposed by Herta Freitag, Roanoke, VA***

For $n \geq 0$, express the larger root of $x^2 - L_n x + (-1)^n = 0$ in terms of α, the larger root of $x^2 - x - |(-1)^n| = 0$.

B-731 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Evaluate the determinant:

$$\begin{vmatrix} F_0 & F_1 & F_2 & F_3 & F_4 \\ F_1 & F_0 & F_1 & F_2 & F_3 \\ F_2 & F_1 & F_0 & F_1 & F_2 \\ F_3 & F_2 & F_1 & F_0 & F_1 \\ F_4 & F_3 & F_2 & F_1 & F_0 \end{vmatrix}$$

Generalize.

B-732 ***Proposed by Richard André-Jeannin, Longwy, France***

Dedicated to Dr. A. P. Hillman

Let (w_n) be any sequence of integers that satisfies the recurrence

$$w_n = pw_{n-1} - qw_{n-2}$$

where p and q are odd integers. Prove that, for all n,

$$w_{n+6} \equiv w_n \pmod 4.$$

B-733 ***Proposed by Piero Filipponi, Rome, Italy***

Write down the Pell sequence, defined by $P_0 = 0, P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$. Form a difference triangle by writing down the successive differences in rows below it. For example,

$$
\begin{array}{ccccccccccccccccc}
0 & & 1 & & 2 & & 5 & & 12 & & 29 & & 70 & & 169 & \cdots \\
& 1 & & 1 & & 3 & & 7 & & 17 & & 41 & & 99 & & \\
& & 0 & & 2 & & 4 & & 10 & & 24 & & 58 & & & \\
& & & 2 & & 2 & & 6 & & 14 & & 34 & & & & \\
& & & & 0 & & 4 & & 8 & & 20 & & & & & \\
& & & & & 4 & & 4 & & 12 & & & & & & \\
& & & & & & 0 & & 8 & & & & & & & \\
& & & & & & & 8 & & & & & & & & \\
& & & & & & & & \ddots & & & & & & &
\end{array}
$$

Identify the pattern that emerges down the left side and prove that this pattern continues.

B-734 ***Proposed by Paul S. Bruckman, Edmonds, WA***

If r is a positive integer, prove that

$$L_{5^r} \equiv L_{5^{r-1}} \pmod{5^r}.$$

B-735 ***Proposed by Curtis Cooper & Robert E. Kennedy, Central Missouri State Asylum for Crazed Mathematicians, Warrensburg, MO***

Let the sequence (y_n) be defined by the recurrence

$$
\begin{aligned}
y_{n+1} = {} & 8y_n + 22y_{n-1} - 190y_{n-2} + 28y_{n-3} + 987y_{n-4} - 700y_{n-5} - 1652y_{n-6} + 1652y_{n-7} \\
& + 700y_{n-8} - 987y_{n-9} - 28y_{n-10} + 190y_{n-11} - 22y_{n-12} - 8y_{n-13} + y_{n-14}
\end{aligned}
$$

for $n \geq 15$ with initial conditions given by the table:

n	y_n
1	1
2	1
3	25
4	121
5	1296
6	9025
7	78961
8	609961
9	5040025
10	40144896
11	326199721
12	2621952025
13	21199651201
14	170859049201
15	1379450250000

Prove that y_n is a perfect square for all positive integers n.

SOLUTIONS

A Sum Involving $F_{2^k}^4$

B-703 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Prove that for all positive integers n,

$$\sum_{k=1}^{n} 4^{n-k} F_{2^k}^4 = \frac{F_{2^{n+1}}^2 - 4^n}{5}.$$

Solution by Bob Prielipp, University of Wisconsin, Oshkosh, WI

Our solution will use the following known result (see Identity I_{39} on page 59 of [1]):

$$(*) \qquad F_m^4 = \frac{F_{2m}^2 - 4(-1)^m F_m^2}{5}.$$

To establish the desired result, it is sufficient to show that

$$\sum_{k=1}^{n} \frac{F_{2^k}^4}{4^k} = \frac{F_{2^{n+1}}^2 - 4^n}{5 \cdot 4^n}.$$

From $(*)$, we have

$$\sum_{k=1}^{n} \frac{F_{2^k}^4}{4^k} = \sum_{k=1}^{n} \frac{F_{2^{k+1}}^2 - 4F_{2^k}^2}{5 \cdot 4^k} = \frac{1}{5} \sum_{k=1}^{n} \left(\frac{F_{2^{k+1}}^2}{4^k} - \frac{F_{2^k}^2}{4^{k-1}} \right)$$

$$= \frac{1}{5} \left(\frac{F_{2^{n+1}}^2}{4^n} - F_2^2 \right) \text{ (by telescoping) } = \frac{F_{2^{n+1}}^2 - 4^n}{5 \cdot 4^n}.$$

The proposer gave the generalization:

$$\sum_{k=1}^{n} 4^{n-k} F_{m2^k}^4 = \frac{F_{m2^{n+1}}^2 - 4^n F_{2m}^2}{5}$$

for all positive integers m and n. The proof is similar. No reader gave any generalizations involving $L_{2^k}^4$. *Apparently there is no closed form for* $\sum_{k=1}^{n} F_{2^k}^4$ *or even* $\sum_{k=1}^{n} F_{2^k}$. *For which constants a, c, r can* $\sum_{k=1}^{n} c^k F_{a^k}^r$ *be expressed in closed form?*

Reference:

1. Verner E. Hoggatt, Jr., *Fibonacci and Lucas Numbers* (Santa Clara, CA: The Fibonacci Association, 1979).

Also solved by Paul S. Bruckman, Herta T. Freitag, C. Georghiou, Russell Jay Hendel, Hans Kappus, Graham Lord, Ray Melham, Blagoj S. Popov, Sahib Singh, and the proposer.

Products of Terms of the Form $ax^2 + by^2$

B-704 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let a and b be fixed integers. Show that if three integers are of the form $ax^2 + by^2$ for some integers x and y, then their product is also of this form.

Solution by Ray Melham, University of Technology, Sydney, Australia

By expanding both sides, it is seen that

$$(ax_1^2+by_1^2)(ax_2^2+by_2^2)(ax_3^2+by_3^2)$$
$$=a(ax_1x_2x_3+bx_1y_2y_3+by_1x_2y_3-by_1y_2x_3)^2+b(ax_1x_2y_3-ax_1y_2x_3-ay_1x_2x_3-by_1y_2y_3)^2.$$

This proves the result.

Flanigan notes that the above identity holds in any commutative ring with identity. The proposer showed that the product of two integers of the form ax^2+by^2 can be written in the form X^2+abY^2 by means of the identity

$$(ax_1^2+by_1^2)(ax_2^2+by_2^2)=(ax_1x_2+by_1y_2)^2+ab(x_1y_2-x_2y_1)^2.$$

He then showed that the product of a number of the form (ax^2+by^2) *and a number of the form X^2+abY^2 can be written in the form (ar^2+bs^2) by means of the identity*

$$(au_1^2+bv_1^2)(u_2^2+abv_2^2)=a(u_1u_2+bv_1v_2)^2+b(u_2v_1-au_1v_2)^2.$$

Also solved by F. J. Flanigan, C. Georghiou, Russell Jay Hendel, Hans Kappus, H.-J. Seiffert, and the proposer. Most of the solutions were similar to that given above.

An Application of a Series Expansion for $(\arcsin x)^2$

B-705 ***Proposed by H.-J. Seiffert, Berlin, Germany***

(a) Prove that $\displaystyle\sum_{n=1}^{\infty}\frac{L_{2n}}{n^2\binom{2n}{n}}=\frac{\pi^2}{5}$.

(b) Find the value of $\displaystyle\sum_{n=1}^{\infty}\frac{F_{2n}}{n^2\binom{2n}{n}}$.

Nearly identical solutions by Russell Euler, Northwest Missouri State University, Maryville, MO; C. Ceorghiou, University of Patras, Patras, Greece; Hans Kappus, Rodersdorf, Switzerland; and Bob Prielipp, University of Wisconsin, Oshkosh, WI.

We start with the known result (see [1], [2], or [3]):

$$\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{n^2\binom{2n}{n}}=2(\arcsin x)^2$$

which converges for $|x|\leq 1$. In particular, for $x=\alpha/2$ and $x=\beta/2$, we have

$$\sum_{n=1}^{\infty}\frac{\alpha^{2n}}{n^2\binom{2n}{n}}=2\left(\arcsin\frac{\alpha}{2}\right)^2 \quad\text{and}\quad \sum_{n=1}^{\infty}\frac{\beta^{2n}}{n^2\binom{2n}{n}}=2\left(\arcsin\frac{\beta}{2}\right)^2.$$

Now, from problem B-674 [*FQ* **29.3** (1991):280], we know that $\cos\pi/5=\alpha/2$ and $\cos 3\pi/5=\beta/2$. This implies that

$$\sin\frac{3\pi}{10}=\sin\left(\frac{\pi}{2}-\frac{\pi}{5}\right)=\cos\frac{3\pi}{5}=\frac{\alpha}{2} \quad\text{and}\quad \sin\left(-\frac{\pi}{10}\right)=\sin\left(\frac{\pi}{2}-\frac{3\pi}{5}\right)=\cos\frac{3\pi}{5}=\frac{\beta}{2}.$$

Thus,

$$\arcsin\frac{\alpha}{2}=\frac{3\pi}{10} \quad\text{and}\quad \arcsin\frac{\beta}{2}=-\frac{\pi}{10}.$$

Therefore,

$$\text{(a)}\qquad \sum_{n=1}^{\infty}\frac{L_{2n}}{n^2\binom{2n}{n}}=2\left[\left(\frac{3\pi}{10}\right)^2+\left(-\frac{\pi}{10}\right)^2\right]=\frac{\pi^2}{5}$$

and

$$\text{(b)}\qquad \sum_{n=1}^{\infty}\frac{F_{2n}}{n^2\binom{2n}{n}}=\frac{2}{\sqrt{5}}\left[\left(\frac{3\pi}{10}\right)^2-\left(-\frac{\pi}{10}\right)^2\right]=\frac{4\sqrt{5}\pi^2}{125}.$$

References:

1. Bruce C. Berndt, *Ramanujan's Notebooks*, Part 1 (New York: Springer Verlag, 1985, p. 262.
2. I. S. Gradshteyn & I. M. Ryzhik, *Tables of Integrals, Series and Products* (New York: Academic Press, 1980), p. 52.
3. L. B. W. Jolley, *Summation of Series*, 2nd ed. rev. (New York: Dover, 1961), p. 146, series 778.

Also solved by Paul S. Bruckman and the proposer.

An Exponential Inequality

B-706 ***Proposed by K. T. Atanassov, Sofia Bulgaria***

Prove that for $n \geq 0$, $\left(\frac{\pi e}{\pi+e}\right)^{1.4n} > F_n$.

Solution by Wray Brady, Chapala, Jalisco, Mexico

Let

$$k=\left(\frac{\pi e}{\pi+e}\right)^{1.4}.$$

We note that $\alpha \approx 1.618$ and $k \approx 1.694$, so that $\alpha < k$. Furthermore, since $\alpha > 1$ and $-1<\beta<0$, we have $|\beta^n| \leq 1 \leq \alpha^n$ for $n \geq 0$. Thus,

$$F_n=\frac{\alpha^n-\beta^n}{\sqrt{5}}\leq\frac{\alpha^n+\alpha^n}{\sqrt{5}}<\frac{2\alpha^n}{2}=\alpha^n<k^n.$$

The proposer also sent in several other inequalities involving Euler's constant and Catalan's constant; however, they were all of the form $k^n > F_n$ where k was some constant larger than α. The conclusion then follows similarly from the fact that $F_n < \alpha^n$. Gilbert showed by taking limits that α is the smallest number with this property. In other words, if $F_n < k^n$ for all $n \geq 0$, then $k \geq \alpha$. Several respondents noted the stronger inequality, $F_n \leq \alpha^{n-1}$ (see page 57 of [1]).

Reference:

1. S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section—Theory and Applications* (Chichester: Ellis Horwood Ltd., 1989).

Also solved by Charles Ashbacher, Glenn Bookhout, Paul S. Bruckman, Joseph E. Chance, C. Georghiou, Peter Gilbert, Pentti Haukkanen, Douglas E. Iannucci, Russell Jay Hendel, Bob Prielipp, Mike Rubenstein, H.-J. Seiffert, Lawrence Somer, Ralph Thomas, and the proposer.

Simple Pythagorean Triple

B-707 *Proposed by Herta T. Freitag, Roanoke, VA*

Consider a Pythagorean triple (a, b, c) such that

$$a = 2\sum_{i=1}^{n} F_i^2 \quad \text{and} \quad c = F_{2n+1},\ n \geq 2.$$

Prove or disprove that b is the product of two Fibonacci numbers.

Solution by H.-J. Seiffert, Berlin, Germany

From equations (I_3) and (I_{11}) of [1], we have $a = 2F_nF_{n+1}$ and $c = F_{n+1}^2 + F_n^2$. Since, in a Pythagorean triple, $b^2 = c^2 - a^2$, we find that

$$b = F_{n+1}^2 - F_n^2 = (F_{n+1} - F_n)(F_{n+1} + F_n) = F_{n-1}F_{n+2},$$

which shows that b is always the product of two Fibonacci numbers.

Reference:

1. Verner E. Hoggatt, Jr., *Fibonacci and Lucas Numbers* (Santa Clara, CA: The Fibonacci Association, 1979).

Also solved by Charles Ashbacher, M. A. Ballieu, Wray Brady, Scott H. Brown, Paul S. Bruckman, Joseph E. Chance, C. Georghiou, Russell Jay Hendel, Joseph J. Kostal, Bob Prielipp, Sahib Singh, Lawrence Somer, Ralph Thomas, and the proposer. Many of the solutions were similar to the featured solution. One solution was received that did not contain the solver's name.

Exponential Summation

B-708 *Proposed by Joseph J. Kostal, University of Illinois at Chicago, IL*

Find the sum of the series $\displaystyle\sum_{k=1}^{\infty} \frac{3^k F_k - 2^k L_k}{6^k}$

Solution 1 by Glenn Bookhout, North Carolina Wesleyan College, Rocky Mount, NC

We use the well-known generating functions for F_n and L_n (see page 53 of [1]). They are given by the equations

(1) $$\sum_{k=0}^{\infty} F_k t^k = \frac{t}{1 - t - t^2}$$

and

(2) $$\sum_{k=0}^{\infty} L_k t^k = \frac{2-t}{1-t-t^2}.$$

Since

$$\lim_{k\to\infty} \frac{F_{k+1}}{F_k} = \alpha$$

by formula (101) of [1], the power series (1) converges for $|t| < 1/\alpha$ by the Ratio Test. Similarly, since

$$\lim_{k\to\infty} \frac{L_{k+1}}{L_k} = \alpha$$

the power series (2) also converges for $|t| < 1/\alpha$.

Substituting 1/2 for t in power series (1) gives

(3) $$\sum_{k=1}^{\infty} \frac{F_k}{2^k} = 2.$$

Substituting 1/3 for t in power series (2) gives $\sum_{k=0}^{\infty}(L_k/3^k) = 3$ so

(4) $$\sum_{k=1}^{\infty} \frac{L_k}{3^k} = 1.$$

It follows from equations (3) and (4) that

$$\sum_{k=1}^{\infty} \frac{3^k F_k - 2^k L_k}{6^k} = 1.$$

Seiffert and Bruckman proceeded similarly, but used the power series

$$\sum_{k=1}^{\infty} L_k t^k = \frac{t(1+2t)}{1-t-t^2}, \quad |t| < \alpha^{-1}.$$

Several readers blindly substituted values into equations (1) and (2) without first noting the radius of convergence of these series.

Reference:

1. S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section—Theory and Applications* (Chichester: Ellis Horwood Ltd., 1989).

Solution 2 by C. Georghiou, University of Patras, Greece

We have the following (converging) geometrical series:

$$\sum_{k=1}^{\infty} \frac{3^k \alpha^k}{6^k} = \frac{\alpha}{2-\alpha} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{3^k \beta^k}{6^k} = \frac{\beta}{2-\beta}.$$

Using the Binet form, $F_k = (\alpha^k - \beta^k)/(\alpha-\beta)$, we get

$$\sum_{k=1}^{\infty} \frac{3^k F_k}{6^k} = \frac{1}{\alpha-\beta}\left[\frac{\alpha}{2-\alpha} - \frac{\beta}{2-\beta}\right] = 2$$

where we have simplified by using the identities $\alpha+\beta = 1$ and $\alpha\beta = -1$.

In the same way, from

$$\sum_{k=1}^{\infty} \frac{2^k \alpha^k}{6^k} = \frac{\alpha}{3-\alpha} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{2^k \beta^k}{6^k} = \frac{\beta}{3-\beta}$$

and the Binet form, $L_k = \alpha^k + \beta^k$, we get

$$\sum_{k=1}^{\infty} \frac{2^k L_k}{6^k} = \frac{\alpha}{3-\alpha} + \frac{\beta}{3-\beta} = 1.$$

Therefore, the given sum evaluates to $2 - 1 = 1$.

Solution 3 by W. R. Spickerman, R. N. Joyner, & R. L. Creech (jointly), East Carolina University, Greenville, NC

Let $S_1 = \sum_{k=1}^{\infty} \frac{F_k}{2^k}$ and $S_2 = \sum_{k=1}^{\infty} \frac{L_k}{3^k}$.

Both series are seen to converge by the Ratio Test. Hence, the series consisting of the differences of successive terms of these series converges to $S_1 - S_2$. That is,

$$\sum_{k=1}^{\infty} \frac{3^k F_k - 2^k L_k}{6^k} = \sum_{k=1}^{\infty} \frac{F_k}{2^k} - \sum_{k=1}^{\infty} \frac{L_k}{3^k} = S_1 - S_2.$$

Multiplying the series for S_1 by 1, 1/2, and 1/4, respectively, we find that

$$\left(1 - \frac{1}{2} - \frac{1}{4}\right) S_1 = \frac{F_1}{2} + \frac{F_2}{4} - \frac{F_1}{4} + \sum_{k=3}^{\infty} \frac{1}{2^k}(F_k - F_{k-1} - F_{k-2}).$$

Since the Fibonacci sequence satisfies the recurrence $F_k = F_{k-1} + F_{k-2}$, the summation in this last equation is 0. Therefore,

$$\frac{1}{4} S_1 = \frac{F_1 + F_2}{4} = \frac{2}{4},$$

so $S_1 = 2$. Similarly,

$$\frac{5}{9} S_2 = \frac{2L_1 + L_2}{9} = \frac{5}{9},$$

so $S_2 = 1$. Hence, the desired sum is $S_1 - S_2 = 2 - 1 = 1$.

Redmond generalized by showing that for sequences defined by $P_n = aP_{n-1} - bP_{n-2}$ and $Q_n = aQ_{n-1} - bQ_{n-2}$ (with $a^2 \neq 4b$), and real numbers A, B, and C, we have

$$\sum_{k=0}^{\infty} \frac{A^k P_k + B^k Q_k}{C^k} = C \left[\frac{c_0(C - A\beta) + c_1(C - A\alpha)}{(C - A\alpha)(C - A\beta)} + \frac{d_0(C - B\beta) + d_1(C - B\alpha)}{(C - B\alpha)(C - B\beta)} \right]$$

where α and β are the roots of the characteristic equation, $x^2 - ax + b = 0$, chosen so that $\alpha - \beta = \sqrt{a^2 - 4b}$ and with initial conditions such that the Binet forms are $P_n = c_0 \alpha^n + c_1 \beta^n$ and $Q_n = d_0 \alpha^n + d_1 \beta^n$. The series converges if $\max(|A\alpha/C|, |A\beta/C|, |B\alpha/C|, B\beta/C|) < 1$.

Also solved by Wray Brady, Scott H. Brown, Paul S. Bruckman, Joseph E. Chance, Russell Euler, Herta T. Freitag (2 solutions), Douglas E. Iannucci, Russell Jay Hendel, Bob Prielipp, Don Redmond, H.-J. Seiffert, Sahib Singh, Ralph Thomas (2 solutions), and the proposer.