FIBONACCI CUBES—A CLASS OF SELF-SIMILAR GRAPHS

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1. INTRODUCTION

The Fibonacci cube [6] is a new class of graphs that are inspired by the famous numbers. Because of the rich properties of the Fibonacci numbers [1], the graph also shows interesting properties. For a graph with N nodes, it is known [6] that the diameter, the edge connectivity, and the node connectivity of the Fibonacci cube are in the order of $O(\log N)$, which are similar to the Boolean cube (or hypercube, n-cube, cosmic cube [9]). A possible application of the Fibonacci cube is in the interconnection of large-scale multi-computers or distributed networks. Here we show that the Fibonacci cube has attractive recurrent structures (called *self-similarity*, §2) in the following sense:

1. A Fibonacci cube can be decomposed into subgraphs which are also Fibonacci cubes by themselves;
2. By suitably defining equivalence classes of vertices in the Fibonacci cube and merging the edges between vertices in different classes in a natural fashion, the resulting graph (of the equivalence classes) is again a Fibonacci cube.

This structural recurrence is useful to derive (divide-and-conquer) algorithms for a parallel computer based on the Fibonacci cube [6]. It is also useful to derive the embeddings of other types of graphs [8]. (See also §4 for discussions.)

This paper is organized as follows. Section 2 defines the Fibonacci cube based on the Fibonacci representation of integers. Section 3 provides a characterization of the new graph and discusses various decompositions. Section 4 briefly summarizes the results that are presented and discusses possible applications. The rest of Section 1 lists notations to be used throughout this paper.

A graph G is a pair (V, E), where V denotes the set of *vertices* (or, alternatively, *nodes*) and E the set of edges. The following terminology and notations will be used [3]:

- We write $G_2 \subseteq G_1$ (or, alternatively, $G_1 \supseteq G_2$) if G_2 is a subgraph of G_1. Write $G_1 \cong G_2$ if the two graphs are *isomorphic*.
- A subgraph of a graph $G = (V, E)$ *induced* by a subset of its vertices, $V' \subseteq V$, is the graph (V', E'). where $E' = \{(i, j) \in E : i, j \in V'\}$
- We write $G_1 \cup G_2$ to denote the graph $(V_1 \cup V_2, E_1 \cup E_2)$, and $G_1 \cap G_2$ to denote $(V_1 \cap V_2, E_1 \cap E_2)$, and $\bigcup_{1=i}^{m} G_i = G_1 \cup G_2 \cup \cdots \cup G_m$.
- If $G_2 \cap G_3 = (\phi, \phi)$, i.e., they are disjoint, then we write $G_1 = G_2 \uplus G_3$ instead of $G_2 \cup G_3$) to emphasize that G_1 consists of two disjoint subgraphs. Also, for convenience, write $\biguplus_{1=i}^{m} G_1 = m \cdot G$ if the graphs are all isomorphic, i.e., $G_i \cong G$ for $1 \leq i \leq m$.

2. DEFINITION OF FIBONACCI CUBE

The Fibonacci cube can be defined by using the Fibonacci representation of integers.

Definition: Assume that i is an integer, and $0 \le i < F_n$, where $n \ge 3$. The *order-n Fibonacci code* (or, simply, *Fibonacci code*, if n is implicit) of i is a sequence of $n-2$ binary digits $(b_{n-1}, \ldots, b_3, b_2)_F$, where

1. $b_j \cdot b_{j+1} = 0$ for $2 \le j \le (n-2)$, and

2. $i = \sum_{j=2}^{n-1} b_j \cdot F_j$.

Example: By *Zeckendorf's theorem* [10], any natural number can be uniquely represented in its Fibonacci code. The Fibonacci representation of an integer $N > 0$ can be obtained by using the following *greedy* approach [4]. First find the greatest F_k that is less than or equal to N, assign a "1" to the bit that corresponds to F_k, then proceed recursively for $N - F_k$ until the remainder is 0. The unassigned bits are 0's. Here the integers from 1 to 20 are given in this notation:

$0 = (000000)_F$, $1 = (000001)_F$, $2 = (000010)_F$, $3 = (000100)_F$, $4 = (000101)_F$, $5 = (001000)_F$,
$6 = (001001)_F$, $7 = (001010)_F$, $8 = (010000)_F$, $9 = (010001)_F$, $10 = (010010)_F$, $11 = (010100)_F$,
$12 = (010101)_F$, $13 = (100000)_F$, $14 = (100001)_F$, $15 = (100010)_F$, $16 = (100100)_F$, $17 = (100101)_F$,
$18 = (101000)_F$, $19 = (101001)_F$, $20 = (101010)_F$.

Remarks: Notice that in the Fibonacci code, the rightmost bit corresponds to F_2. rather than F_1. Note also that no consecutive 1's appeared in the Fibonacci codes; to represent a number between 0 and $F_n - 1$ requires $n-2$ bits. Therefore, to represent the number $21 = (1000000)_F$ requires an additional bit (cf. the preceding example). □

Let $I = (b_{n-1}, \ldots, b_3, b_2)$ and $J = (c_{n-1}, \ldots, c_3, c_2)$ denote two sequences of 0's and 1's. The *Hamming distance* between I and J, denoted by $H(I, J)$, is the number of bits where the two sequences differ.

Definition [Fibonacci Cube of Order n]: Let $F(i)$ denote the Fibonacci code of i. The *Fibonacci cube of order n*, denoted by Γ_n, is a graph (V_n, E_n), where $V_n = \{0, 1, \ldots, F_n - 1\}$ and $E_n = \{(i, j) : H(F(i), F(j)) = 1,\ 0 \le i, j \le F_n - 1\}$. Define $\Gamma_0 = (\phi, \phi)$. □

Figure 1 shows the Fibonacci cubes Γ_i for $1 \le i \le 7$.

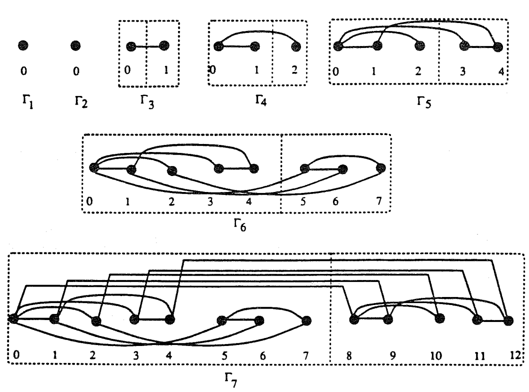


FIGURE 1. Fibonacci Cubes

Remarks:

1. The definition of Fibonacci cube parallels that of Boolean cube (hypercube). Specifically, the *Boolean cube of dimension n*, denoted by B_n, is a graph (V_n, E_n), where $V_n = \{0, 1, \ldots, 2^n - 1\}$ and $(i, j) \in E_n$ if and only if $H(I_B, J_B) = 1$, where I_B and J_B denote the (ordinary) binary representation of i and j, $0 \le i,\ j \le 2^n - 1$ (Fig. 2).

2. The preceding definition of the Fibonacci cube can be modified to accommodate a Fibonacci cube of *size* (i.e., number of nodes) N for an arbitrary integer $N \ge 1$ [6]. However, as we will see, when the size of the cube is a Fibonacci number, the Fibonacci cube has a recurrent structure and hence is more desirable.

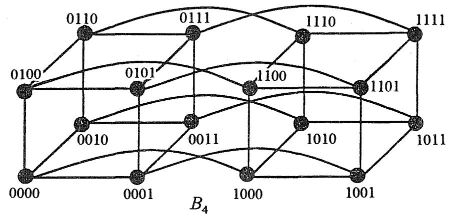


FIGURE 2. A Boolean Cube

3. RECURSIVE DECOMPOSITIONS OF THE FIBONACCI CUBE

In [6] it is shown that the *Fibonacci cube of order n*, where $n \ge 2$, contains two disjoint subgraphs that are isomorphic to Γ_{n-1} and Γ_{n-2}, respectively (with proper renaming of the vertices in Γ_{n-2}); moreover, there are exactly F_{n-2} edges linking the two subgraphs together.

Theorem 1 (Characterization of the Fibonacci Cube): Let $\Gamma_n = (V_n, E_n)$ denote the Fibonacci cube of order n, where $n \ge 2$. Let LOW(n) (resp., HIGH (n)) denote the subgraph induced by the set of nodes in $\{0, 1, \ldots, F_{n-1} - 1\}$ (resp., $\{F_{n-1}, \ldots, F_n - 1]$). Then

1. $\mathrm{LOW}(n) \cong \Gamma_{n-1}$ and $HIGH(n) \cong \Gamma_{n-2}$;

2. Let $0 \le i \le F_{n-1} - 1$ and $F_{n-1} \le j \le F_n - 1$. $(i, j) \in \Gamma_n$ if and only if $j - i = F_{n-1}$.

Proof: (We refer to [6].) □

Example: (Fig. 1). Γ_6 can be decomposed into two subgraphs that are isomorphic to Γ_5 and Γ_4, respectively. There are $F_4 = 3$ edges connecting the two subgraphs.

The above characterization can be expressed in terms of Fibonacci codes.

Corollary 1: Assume that $n \ge 2$. Let G_0 (resp., G_1) denote the subgraph of Γ_n induced by the set of vertices $\{i : i = (0b_{n-2}b_{n-3} \ldots b_2)_F\}$ (resp., $\{j : j = (1b_{n-2}b_{n-3} \ldots b_2)_F\}$). Then

1. $G_0 \cong \Gamma_{n-1}$ and $G_1 \cong \Gamma_{n-2}$.

2. Let $i = (0b_{n-2}b_{n-3} \ldots b_2)_F \in G_0$ and $j = (1b'_{n-1}b'_{n-2} \ldots b'_2)_F \in G_1$. $(i, j) \in \Gamma_n$ if and only if $b_k = b'_k$ for $n - 2 \ge k \ge 2$.

Proof (outlines):

Statement 1: Let $i = (b_{n-1} \ldots b_2)_F$. There are two cases:

1. $0 \le i < F_{n-1}$, in this case $b_{n-1} = 0$.

2. $F_{n-1} \le i < F_n$, in this case $b_{n-1} = 1$.

The result then follows by observing that $G_0 = \text{LOW}(n)$ and $G_1 = \text{HIGH}(n)$.

Statement 2 follows from Statement 2 of Theorem 1. □

3.1 A Generalization

A generalization of Theorem 1 can be obtained by applying the decomposition recursively. Recall that $\biguplus_{1=i}^{m} G_i = m \cdot G$ if $G_i \cong G$ for all $1 \le i \le m$.

Example: In Figure 1 we see that Γ_6 contains a subgraph Γ_5 and a subgraph Γ_4 (after renaming vertices). Since Γ_5 can be decomposed into a subgraph Γ_4 and a subgraph Γ_3, so Γ_6 contains two disjoint Γ_4 and one Γ_3. Using the notations introduced, we will write $\Gamma_6 \supseteq (2 \cdot \Gamma_4 \uplus \Gamma_3)$.

Theorem 2: Assume that $2 \le k \le n$. The Fibonacci cube of order n (Γ_n) admits the following decompositions:

(a) $\Gamma_n \supseteq (F_k \cdot \Gamma_{n-k+1} \uplus F_{k-1} \cdot \Gamma_{n-k})$;

(b) $\Gamma_n \supseteq (F_{n-k+1} \cdot \Gamma_k \uplus F_{n-k} \cdot \Gamma_{k-1})$.

Proof: We will prove Statement (a) by induction on n.

(Basis) If $n = 2$, then $k = 2$ and the statement can be easily verified.

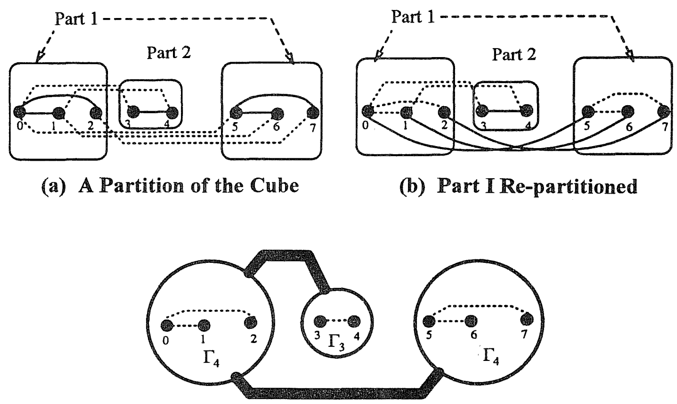
(Hypothesis) Assume that the statement is true for $n \le N$.

(Induction) Consider the case $n = N+1$. By Theorem 1, Γ_{N+1} consists of one Γ_N and one Γ_{N-1}. By hypothesis, for any k between 2 and $N-1$, Γ_N (resp., Γ_{N-1}) may be divided into F_k copies of Γ_{N-k+1} and F_{k-1} copies of Γ_{N-k} (resp., F_{k-1} copies of $\Gamma_{(N-1)-(k-1)+1}$ and F_{k-2} copies of $\Gamma_{(N-1)-(k-1)}$). Together, the number of copies of $\Gamma_{(N+1)-(k+1)+1}$ is $F_k + F_{k-1} = F_{k+1}$ and the number of copies of $\Gamma_{(N+1)-(k+1)}$ is $F_{k-1} + F_{k-2} = F_k = F_{(k+1)-1}$, which completes the proof in the case $3 \le k \le N$. The case $k = 2$, $N+1$ can be easily verified.

Statement (b) can be proved similarly [6]. □

Remarks: Note that the decompositions listed in the preceding lemma are based on the following property of Fibonacci numbers: $F_n = F_k F_{n-k+1} + F_{k-1} \cdot F_{n-k}$, which holds true for *all* integers k and n [4]. In Theorem 2, the first term $F_k \cdot F_{n-k+1}$ corresponds to a subgraph of Γ_n which is either divided into (i) F_k copies of Γ_{n-k+1} or (ii) F_{n-k+1} copies of Γ_k. (The second term $F_{k-1} \cdot F_{n-k}$ also suggests two possible decompositions.) Note that the decomposition in (ii) can be derived from (i) as follows. [Constructing (i) from (ii) is similar.] Each subgraph Γ_k of (ii) is essentially constructed by taking one node from each of the F_k copies of Γ_{n-k+1} in (i). By construction, no two subgraphs from the two decompositions in (i) and (ii) share more than one node. Such decompositions will be referred to as *orthogonal decompositions.* □

Example: Take Γ_6 (Fig. 3) for instance. Let $k = 3$ and note that $8 = F_6 = F_3 \cdot F_4 + F_2 \cdot F_3 = 2 \cdot 3 + 1 \cdot 2$. By Theorem 2, Γ_6 can be decomposed into (Part 1) two copies of Γ_4 and (Part 2) one copy of Γ_3. In Figure 3(a), Part 1 consists of two subgraphs whose vertex sets are, respectively, $\{0, 1, 2\}$ and $\{5, 6, 7\}$. Note that, by Theorem 2, an alternative (and orthogonal) decomposition of Part 1 would be to divide the same set of nodes into three copies of Γ_3, where each Γ_3 is formed by taking one node from each copy of Γ_4. In Figure 3(b), for example, the nodes in Part 1 are re-partitioned into the following sets $\{0, 5\}$, $\{1, 6\}$, and $\{2, 7\}$. Notice that no two subgraphs from the first partition and the second partition share more than one common node. Thus, the two partitions of Part 1 are orthogonal. Similarly, nodes in Part 2 can be redivided into two copies of Γ_1. □



(a) A Partition of the Cube (b) Part I Re-partitioned

(c) Deriving a Quotient Fibonacci Cube

FIGURE 3. Decomposition of a Fibonacci Cube

In terms of the Fibonacci codes, we have the following corollary.

Corollary 2: Assume that $n \geq 2$. Let k and d denote two integers, where $1 \leq k \leq n-2$ and $0 \leq d \leq F_{n-k} - 1$. Let $G_n(k, d)$ denote the subgraph of Γ_n induced by the set of vertices in $\{(b_{n-1}b_{n-2} \ldots b_2)_F : (b_{n-1}b_{n-2} \ldots b_{n-k})_F = d\}$. Then

1. $G_n(k, d) \cong \Gamma_{n-k}$ if $b_{n-k} = 0$, and

2. $G_n(k, d) \cong \Gamma_{n-k-1}$ if $b_{n-k} = 1$.

Proof: The argument parallels that of Theorem 2 (replace all instances of Theorem 1 with Corollary 1) [8]. □

3.2 Quotient Fibonacci Cubes

We will identify another level of recurrence with the decompositions of the Fibonacci cube in which the graph Γ_n is scaled down to a smaller Fibonacci cube. In fact, for any Γ_n and any integer k, where $1 \leq k \leq n-2$, we can define a *Quotient Fibonacci Cube* Γ_n / k as described in the following. (See [2]; cf. Theorem 2.)

We describe the idea in intuitive terms followed by a formal definition. Consider the first decomposition [i.e., Decomposition (a)] listed in Theorem 2. Let each of the $F_{k+1} = F_k + F_{k-1}$ subgraphs (Γ_{n-k+1} or Γ_{n-k}) be considered as an equivalence class. Then each node v in Γ_n / k corresponds to such an equivalence class. The edges between two equivalence classes v_1 and v_2 are given by $(v_1, v_2) \in \Gamma_n / k$ if and only if $\{(v_1, v_2) : v_1 \in v_1$ and $v_2 \in v_2\} \neq \phi$. (In other words, the edges connecting nodes in two subgraphs are merged into one.) Then the resulting graph Γ_n / k of the equivalence classes is itself a Fibonacci cube (as will be proved). A similar observation applies to Decomposition (b).

Example: Consider Γ_6 again [cf. Fig. 3(c)]. By considering each of the two copies of Γ_4 (indicated as Part 1 in Fig. 3) and the Γ_3 (Part 2) as a single node (an equivalence class) then merging the edges connecting them (as described in the preceding remarks), the resulting graph is isomorphic to Γ_4.

Definition Assume that $1 \le k \le n-2$ and $0 \le d \le F_{k+2}-1$. Let $G_n(k,d)$ denote the subgraph of Γ_n induced by the set of vertices in $\{(b_{n-1}b_{n-2}\dots b_2)_F : (b_{n-1}b_{n-2}\dots b_{n-k})_F = d\}$. Then the *Quotient Fibonacci Cube* $\Gamma_n / k = (V_n / k, E_n / k)$ is given by:

1. $(V_n / k = \{G_n(k,d) : 0 \le d \le F_{k+2}-1\}$, and

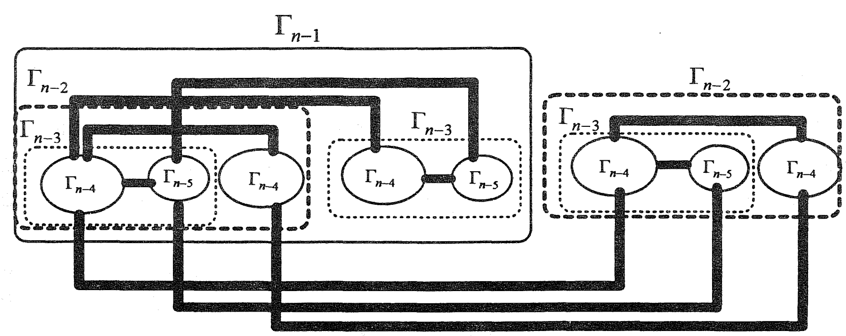
2. $(G_n(k,d),\ G_n(k,d')) \in E_n / k$ if and only if $d \ne d'$ and $\{(v_1, v_2) \in \Gamma_n : v_1 \in G_n(k,d),\ v_2 \in G_n(k,d')\} \ne \phi$

Theorem 3: Let Γ_n / k be the quotient Fibonacci cube as defined before, where $1 \le k \le n-2$. Then $\Gamma_n / k \cong \Gamma_{k+2}$.

Proof (outlines): It is straightforward to verify the theorem for $k = 1$. For example, in Theorem 1 (which corresponds to the case in which $k = 1$), the vertices in LOW(n) [resp., HIGH (n)] can be taken as an equivalence class v_1 (resp., v_2), and edges connecting LOW(n) and HIGH(n) can be taken as a single edge (v_1, v_2). The resulting graph $\Gamma_n / 1 = (\{v_1, v_2\}, \{(v_1, v_2)\})$ is isomorphic to Γ_3.

The general case can be proved inductively by noting that each of the subgraphs can be decomposed recursively and there are links between these subgraphs (Theorem 1). □

Example: Figure 4 shows that $\Gamma_n / 4$ can be derived from Γ_n in four refining steps. In the first step (when $k = 1$) decomposing Γ_n into (a) Γ_{n-1} and (b) Γ_{n-2}. By interpreting the edges between Γ_{n-1} and Γ_{n-2} as a single edge, the resulting graph is $\Gamma_n / 1$, which is isomorphic to Γ_3 (cf. Fig. 1). In the subsequent steps (k = 2, 3, 4), Part (a) and Part (b) are recursively decomposed. The resulting graph $\Gamma_n / 4$ is isomorphic to Γ_6 (cf. Fig. 1).



$\Gamma_{n/4} \cong \Gamma_6$

FIGURE 4. A Quotient Fibonacci Cube

3.3 Other Recursive Decompositions of Fibonacci Cubes

There are other conceivable ways to decompose the Fibonacci cube. We list the following while omitting the details of their proofs.

Lemma 1: Let Γ_n denote the Fibonacci cube of order n. Assume that $n \geq 1$. Then

(a) $\Gamma_{2n} \supseteq (G_1' \uplus G_3' \uplus G_5' \uplus \cdots \uplus G_{2n-1}')$, where $G_k' \cong \Gamma_k$ for $k = 1, 3, 5, \ldots, (2n-1)$.

(b) $\Gamma_{n+2} \supseteq (G_1' \uplus G_2' \uplus G_3' \uplus \cdots \uplus G_n')$, where $G_k' \cong \Gamma_k$ for $k = 1, 2, 3, \ldots, n$.

Proof (outlines): Part (a) of this lemma is inspired by the known recurrence of Fibonacci numbers: $F_{2n} = \sum_{1 \leq k \leq n} F_{2k-1}$. Specifically, the graph Γ_{2n} can be decomposed into a copy of Γ_{2n-1} and a copy of Γ_{2n-2}. The latter can be decomposed further into Γ_{2n-3} and Γ_{2n-4}. Decompose Γ_{2n-4} again and we have Γ_{2n-5} and so on.

Part (b) is based on $F_{n+2} = \sum_{1 \leq k \leq n} F_k + 1$. □

Example Again consider Γ_6. Since $6 = 2 \cdot 3$, by Lemma 1(a), it can be decomposed into three subgraphs: Γ_1, Γ_3, and Γ_5. Also, since $6 = 4 + 2$, by Lemma 1(b), it can be decomposed into one Γ_1, one Γ_2, one Γ_3, and one Γ_4, and all of the subgraphs are disjoint.

4. DISCUSSION AND CONCLUSION

A possible application of the Fibonacci cube is in the interconnection of large-scale multicomputers, where a node corresponds to a processor and an edge to a communication link. In [6], it is shown that the Fibonacci cube contains about 1/5 fewer edges than the Boolean cube for the same number of vertices. Considering the relative sparsity in connections and the asymmetry in structure, it may well be expected that the Fibonacci cube cannot be as flexible as the Boolean cube, and certain functionality may be lost. For example, in the context of interconnection networks, the communication delays may become greater than that based on the Boolean cube, and the power of *embedding* (i.e., emulating other types of graphs) may be inferior to the Boolean cube. Nevertheless, because of the rich properties of Fibonacci numbers, we have been able to show here that the Fibonacci cubes can be flexibly decomposed into subgraphs of same kind (we are tempted to call this property *self-similarity*). In [8], by using these recursive decompositions, it is shown that the Fibonacci cube is flexible enough to embed common graphs such as linear arrays, rings, certain kinds of meshes, tori (mesh with wraparound), and trees, all with perfect dilation and expansion.

The recursive nature of the Fibonacci cube also has implications to the design and analysis of algorithms for parallel computers that are based on the Fibonacci cube. For example, to find the sum (product, maximum, and other associative operations) of a sequence of numbers, the data items can be distributed on the nodes (processors) of the Fibonacci cube. The sum can be found in a divide-and-conquer fashion, which matches well with the recursive decomposition of the graph. In [6], by using this approach, several routing algorithms have been designed for computer architectures based on the Fibonacci cube.

Perhaps the most interesting (and plausible) application of the self-similarity is in fault-tolerant computing. Again consider a parallel computer based on the Fibonacci cube. When some links or nodes of the computer fail, other functioning links and nodes may still be reconfigured to a smaller (but similar) graph and continue to operate (albeit with a degraded performance). In a multiple-processor system, one can also take advantage of this self-similarity to allocate processing resources to multiple users (each user could be assigned a subcube of some size).

We call for further investigation of this new class of graphs.

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AMS Classification Numbers: 68M10, 68Q22, 90B12

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A NEW BOOK ON LIBER ABACI

The editor has recently been informed that a new book on Fibonacci's *Liber Abaci* has appeared in Germany. The editor has been told that the book was written by Professor Heinz Lüneburg, a mathematics professor at the University of Kaiserslautern. The book's title was said to be LEONARDI PISANI LIBER ABACI ODER LESEVERGNÜGEN EINES MATHEMATIKERS. The publisher was reported as BI Verlag, Mannheim, and the cost was said to be 68 Deutsch Marks.