RATIONAL NUMBERS WITH NON-TERMINATING, NON-PERIODIC MODIFIED ENGEL-TYPE EXPANSIONS

Jeffrey Shallit
Department of Computer Science, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada
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(Submitted April 1991)

In a recent paper [3] Kalpazidou, Knopfmacher, & Knopfmacher discussed expansions for real numbers of the form

\[ A = a_0 + \frac{1}{a_1} + \frac{1}{a_1 + 1} \cdot \frac{1}{a_2} + \frac{1}{a_2 + 1} \cdot \frac{1}{a_3} + \cdots \]

which they called a "modified Engel-type" alternating expansion. Here \( a_0 \) is an integer and \( a_i \) is a positive integer for \( i \geq 1 \). If \( a_{i+1} \geq a_i \), this expansion is essentially unique. To save space, we will abbreviate (1) by \( A = \{a_0, a_1, a_2, \ldots \} \).

They say, "The question of whether or not all rationals have a finite or recurring expansion has not been settled." (By "recurring" we understand "ultimately periodic.")

In this note, we prove that the rational numbers \( \frac{2}{2r+1} \) (\( r \) an integer \( \geq 2 \)) have modified Engel-type expansions that are neither finite nor ultimately periodic.

Theorem: Let \( r \) be an integer \( \geq 1 \). Then

\[ \frac{2}{2r+1} = \{a_0, a_1, a_2, \ldots \} \]

where \( a_0 = 0 \), and \( a_{2i-1} = b_i \), \( a_{2i} = 2b_i - 1 \) for \( i \geq 1 \), and \( b_1 = r \), \( b_{n+1} = 2b_n^2 - 1 \) for \( n \geq 1 \).

Proof: As in [3], we have

\[ a_0 = \lfloor A \rfloor, \quad A_1 = A - a_0, \quad a_n = \lfloor 1/A_n \rfloor \text{ for } n \geq 1, \text{ and} \]

\[ A_{n+1} = (1/a_n - A_n)(a_n + 1) \text{ for } n \geq 1. \]

From this we see that \( a_0 = \lfloor \frac{2}{2r+1} \rfloor = 0 \).

We now prove the following four assertions by induction on \( n \): (i) \( A_{2n-1} = \frac{2}{2b_n \pm 1} \); (ii) \( a_{2n-1} = b_n \); (iii) \( A_{2n} = \frac{b_{n+1}}{b_n(2b_n \pm 1)} \); and (iv) \( a_{2n} = 2b_n - 1 \).

It is easy to verify these assertions for \( n = 1 \), as we find

(i) \( A_1 = \frac{2}{2+1} = \frac{2}{2b_1 + 1} \);

(ii) \( a_1 = \lfloor \frac{1}{A_1} \rfloor = r = b_1 \);

(iii) \( A_2 = (1 - \frac{2}{2r+1})(r+1) = \frac{r+1}{r+1} = \frac{b_{n+1}}{b_n(2b_n \pm 1)} \);

(iv) \( a_2 = \lfloor \frac{1}{A_2} \rfloor = \lfloor \frac{r(2r+1)}{r+1} \rfloor = \lfloor 2r - 1 + \frac{1}{r+1} \rfloor = 2r - 1 + 2b_1 - 1 \).

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Now assume the result is true for all \( i \leq n \). We prove it for \( n + 1 \):

(i) \[ A_{2n+1} = \left( \frac{1}{a_{2n}} - A_{2n} \right)(a_{2n} + 1) = \left( \frac{1}{2b_n - 1} - \frac{b_n + 1}{b_n(2b_n + 1)} \right)(2b_n) = \frac{2}{4b_n^2 - 1} = \frac{2}{2b_{n+1} + 1}. \]

(ii) \[ a_{2n+1} = \left[ \frac{1}{A_{2n+1}} \right] = \left[ \frac{2b_{n+1} + 1}{2} \right] = b_{n+1}. \]

(iii) \[ A_{2n+2} = \left( \frac{1}{a_{2n+1}} - A_{2n+1} \right)(a_{2n+1} + 1) = \left( \frac{1}{b_{n+1}} - \frac{2}{2b_{n+1} + 1} \right)(b_{n+1} + 1) = \frac{b_{n+1} + 1}{b_{n+1}(2b_{n+1} + 1)}. \]

(iv) \[ a_{n+2} = \left[ \frac{1}{A_{2n+2}} \right] = \left[ \frac{b_{n+2}(2b_{n+1} + 1)}{b_{n+1} + 1} \right] = 2b_{n+1} - 1 + \left( \frac{1}{b_{n+1} + 1} \right) = 2b_{n+1} - 1. \]

This completes the proof. \( \square \)

**Corollary:** For \( r \geq 2 \), the rational numbers \( \frac{2}{2r+1} \) have non-terminating, non-ultimately-periodic modified Engel-type expansions.

**Additional Remarks:**

- For \( r = 1 \), the theorem gives the ultimately periodic expansion
  \[ \frac{2}{3} = \{0, 1, 1, 1, 1, \ldots \}. \]
- For \( r \geq 2 \), the expansion is not ultimately periodic; e.g.,
  \[ \frac{2}{5} = \{0, 2, 3, 7, 13, 97, 193, 18817, \ldots \}. \]

In this case, we have the following table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( a_n )</th>
<th>( b_n )</th>
<th>( A_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2/5</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>3/10</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>97</td>
<td>2/15</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
<td>18817</td>
<td>8/105</td>
</tr>
<tr>
<td>5</td>
<td>97</td>
<td>708158977</td>
<td>2/195</td>
</tr>
<tr>
<td>6</td>
<td>193</td>
<td>1002978273411373057</td>
<td>89/18915</td>
</tr>
</tbody>
</table>

- The sequence \( b_1, b_2, \ldots = 2, 7, 97, 18817, 708158977, \ldots \), corresponding to \( r = 2 \), appears to have been discussed first by G. Cantor in 1869 [1], who gave the infinite product
  \[ \sqrt{3} = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{7}\right)\left(1 + \frac{1}{97}\right) \ldots. \]

For more on this product of Cantor, see Spiess [9], Sierpinski [7], Engel [2], Stratemeyer [10, 11], Ostrowski [6], and Mendès France & van der Poorten [5]. The sequence 2, 7, 97, 18817, ... was also discussed by Lucas [4]. It is sequence #720 in Sloane [8].

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- The sequence $b_1, b_2, \ldots = 3, 17, 577, 665857, \ldots$, corresponding to $r = 3$, was also discussed by Cantor [1], who gave the infinite product

\[ \sqrt{2} = \left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{17}\right)\left(1 + \frac{1}{577}\right) \ldots. \]

Also see the papers mentioned above. The sequence was also discussed by Wilf [12], and it is sequence #1234 in Sloane [8].

- It is easy to prove that $b_{n+1} = B_n^2$, where $B_0 = 1, B_1 = r$, and $B_n = 2rB_{n-1} - B_{n-2}$ for $n \geq 2$. This gives a closed form for the sequence $(b_n)$:

\[ b_{n+1} = \frac{(r + \sqrt{r^2 - 1})^{2^n} + (r - \sqrt{r^2 - 1})^{2^n}}{2}. \]

- $3/7$ is the "simplest" rational for which no simple description of the terms in its modified Engel-type expansion is known. The first forty terms are as follows:

\[ \frac{3}{7} = \{0, 2, 4, 5, 7, 8, 10, 25, 53, 124, 574, 2431, 13147, 27167, 229073, 315416, 435474, 771789, 1522716, 3833889, 7878986, 7922488, 8844776, 9182596, 9388467, 14781524, 135097360, 1374449987, 1561240840, 4408239956, 11166053604, 12014224315, 23110106464, 553192836372, 90044772231, 1189661630241, 2058097840143484, 6730348855426376, 12928512475357529, \ldots\}. \]

More generally, it would be of interest to know whether it is possible to characterize the modified Engel expansion of every rational number.

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AMS Classification Numbers: 11A67, 11B83, 11B37

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