

THREE-SQUARE THEOREM AS AN APPLICATION OF ANDREWS' IDENTITY

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1. INTRODUCTION

The representation of an integer n as a sum of k squares is one of the most beautiful problems in the theory of numbers. Such representations are useful in lattice point problems, crystallography, and certain problems in mechanics [6, pp. 1-4]. If $r_k(n)$ denotes the number of representations of an integer n as a sum of k squares, Jacobi's two- and four-square theorems [9] are:

$$(1) \quad r_2(n) = 4[d_1(n) - d_3(n)]$$

and

$$(2) \quad r_4(n) = 8 \sum_{\substack{d|n \\ d \equiv 0 \pmod{4}}} d$$

where $d_i(n)$ denotes the number of divisors of n , $d \equiv i \pmod{4}$. In literature there are several proofs of (1) and (2). For instance, M. D. Hirschhorn [7; 8] proved (1) and (2) using Jacobi's triple product identity. S. Bhargava & Chandrashekar Adiga [4] have proved (1) and (2) as a consequence of Ramanujan's ${}_1\Psi_1$ summation formula [10]. Recently R. Askey [2] has proved (1) and also derived a formula for the representation of an integer as a sum of a square and twice a square. The authors [5] have derived a formula for the representation of an integer as a sum of a square and thrice a square. These works of Askey [2] and the authors [5] also rely on Ramanujan's ${}_1\Psi_1$ summation [10].

In 1951 P. T. Bateman [3] obtained the following formula for $r_3(n)$:

$$(3) \quad r_3(n) = \frac{16}{\pi} \sqrt{n} L(1, \chi) q(n) P(n),$$

where

$$n = 4^a n_1, \quad 4 \nmid n_1,$$

$$q(n) = \begin{cases} 0 & \text{if } n_1 \equiv 7 \pmod{8}, \\ 2^{-a} & \text{if } n_1 \equiv 3 \pmod{8}, \\ 3 \cdot 2^{-a-1} & \text{if } n_1 \equiv 1, 2, 5, \text{ or } 6 \pmod{8}, \end{cases}$$

$$P(n) = \prod_{\substack{p^{2b}|n \\ p \text{ odd}}} \left[1 + \sum_{j=1}^{b-1} p^{-j} + p^{-b} \left(1 - \left[\frac{(-n/p^{2b})}{p} \right] \frac{1}{p} \right)^{-1} \right],$$

($P(n) = 1$ for square-free n), and

$L(S, \chi) = \sum_{m=1}^{\infty} \chi(m)m^{-s}$ with $\chi(m)$, the Legendre-Jacobi-Kronecker symbol:

$$\chi(m) = \left(\frac{-4}{m}\right) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod{4}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \\ -1 & \text{if } m \equiv 3 \pmod{4}. \end{cases}$$

In this note we obtain an alternate formula (13) for $r_3(n)$ which involves only partition functions unlike Bateman's formula (3) which is expressed in terms of Dirichlet's series [6, pp. 54, 55]. To derive our formula (13) for $r_3(n)$, we employ G. E. Andrews' [1] generalization of Ramanujan's ${}_1\Psi_1$ summation:

$$(4) \quad \frac{(a^{-1} - b^{-1})(A)_{\infty}(B)_{\infty}(bq/a)_{\infty}(aq/b)_{\infty}(q)_{\infty}(AB/ab)_{\infty}}{(-b)_{\infty}(-a)_{\infty}(-A/b)_{\infty}(-A/a)_{\infty}(-B/b)_{\infty}(-B/a)_{\infty}}$$

$$= a^{-1} \sum_{m=0}^{\infty} \frac{(-q/a)_m (AB/ab)_m (-b)^m}{(-B/a)_{m+1} (-A/a)_{m+1}} - b^{-1} \sum_{m=0}^{\infty} \frac{(A)_m (-aq/B)_m (-B/b)^m}{(-a)_{m+1} (-A/b)_{m+1}},$$

where

$$(a)_{\infty} = (a; q)_{\infty} = \prod_{m=0}^{\infty} (1 - aq^m)$$

and

$$(a)_m = (a; q)_m = \frac{(a; q)_{\infty}}{(aq^m; q)_{\infty}}, \quad |q| < 1.$$

2. THREE-SQUARE THEOREM

In this section we derive a formula for $r_3(n)$, the number of representations of an integer n as a sum of three squares. For convenience, we first transform Andrews' formula (4).

Lemma 2.1 (G. E. Andrews' [1]):

$$(5) \quad \frac{(A; q^2)_{\infty}(-A\beta/\alpha qz; q^2)_{\infty}(-zq; q^2)_{\infty}(-q/z; q^2)_{\infty}(q^2; q^2)_{\infty}(\alpha\beta q^2; q^2)_{\infty}}{(-A/\alpha qz; q^2)_{\infty}(A/\alpha q^2; q^2)_{\infty}(-\alpha qz; q^2)_{\infty}(-\beta q/z; q^2)_{\infty}(\alpha q^2; q^2)_{\infty}(\beta q^2; q^2)_{\infty}}$$

$$= \frac{1}{[1 - (A/\alpha q^2)]} + \sum_{m=1}^{\infty} \frac{(1/\alpha; q^2)_m (-A\beta/\alpha qz; q^2)_m (-\alpha q)^m}{(\beta q^2; q^2)_m (A/\alpha q^2; q^2)_{m+1}} z^m$$

$$+ \sum_{m=1}^{\infty} \frac{(1/\beta; q^2)_m (A; q^2)_{m-1} (-\beta q)^m}{(\alpha q^2; q^2)_m (-A/\alpha qz; q^2)_m} z^{-m}$$

if $|\beta q| < |z| < 1/|\alpha q|$ and $|q| < 1$ with none of the factors in the denominators of (5) being 0.

Proof: Equation (4) is equivalent to

$$\begin{aligned} & \frac{a^{-1}[1-(a/b)](A)_\infty(B)_\infty(bq/a)_\infty(aq/b)_\infty(q)_\infty(AB/ab)_\infty}{[1+(B/a)](-b)_\infty(-a)_\infty(-A/b)_\infty(-A/a)_\infty(-B/b)_\infty(Bq/a)_\infty} \\ &= a^{-1} \frac{1}{[1+(B/a)]} \sum_{m=0}^{\infty} \frac{(-q/a)_m (AB/ab)_m (-b)^m}{(-Bq/a)_m (-A/a)_{m+1}} \\ & \quad - b^{-1} \frac{(-b/B)}{[1+(a/B)]} \sum_{m=0}^{\infty} \frac{(A)_m (-a/B)_{m+1} (-b/B)^{-m-1}}{(-a)_{m+1} (-A/b)_{m+1}} \end{aligned}$$

which, in turn is equivalent to

$$\begin{aligned} (6) \quad & \frac{(A)_\infty(B)_\infty(bq/a)_\infty(a/b)_\infty(q)_\infty(AB/ab)_\infty}{(-b)_\infty(-a)_\infty(-A/b)_\infty(-A/a)_\infty(-B/b)_\infty(-Bq/a)_\infty} \\ &= \sum_{m=0}^{\infty} \frac{(-q/a)_m (AB/ab)_m (-b)^m}{(-Bq/a)_m (-A/a)_{m+1}} + \sum_{m=0}^{\infty} \frac{(A)_m (-a/B)_{m+1} (-b/B)^{-m-1}}{(-a)_{m+1} (-A/b)_{m+1}}. \end{aligned}$$

Change b to $-z$, a to $-q/a'$, B to b'/a' in (6) to obtain

$$\begin{aligned} (7) \quad & \frac{(A)_\infty(b'/a')_\infty(za')_\infty(q/a'z)_\infty(q)_\infty(Ab'/zq)_\infty}{(z)_\infty(q/a')_\infty(A/z)_\infty(Aa'/q)_\infty(b'/a'z)_\infty(b')_\infty} \\ &= \sum_{m=0}^{\infty} \frac{(a')_m (Ab'/zq)_m z^m}{(b')_m (Aa'/q)_{m+1}} + \sum_{m=0}^{\infty} \frac{(A)_m (q/b')_{m+1} (a'z/b')^{-(m+1)}}{(q/a')_{m+1} (A/z)_{m+1}}. \end{aligned}$$

Change q to q^2 , a' to $1/\alpha$, b' to βq^2 , and z to $-\alpha q z$ in (7) to obtain (5). Hence, the lemma.

Corollary 2.1:

$$\begin{aligned} (8) \quad & \left(\sum_{n=-\infty}^{\infty} q^{n^2} \right)^3 = \frac{(-q; q^2)_\infty^3 (q^2; q^2)_\infty^3}{(q; q^2)_\infty^3 (-q^2; q^2)_\infty^3} \\ &= 1 + 2 \sum_{m=1}^{\infty} \frac{(-q; q^2)_m q^m}{(1+q^{2m})(-q^2; q^2)_m} + 4 \sum_{m=1}^{\infty} \frac{(q^2; q^2)_{m-1} q^m}{(1+q^{2m})(q; q^2)_m}. \end{aligned}$$

Proof: Putting $\alpha = \beta = -1$, $z = 1$, and $A = q^2$ in (5), we have the second of the equations (8), the first being a well-known theta-function identity [10]. In fact, put $z = 1$, $A = \alpha = \beta = 0$ in (5) and use the easily verified Euler identity

$$(-q; q^2)_\infty = 1/(q; q^2)_\infty (-q^2; q^2)_\infty.$$

Before stating the main theorem of this section, we introduce two partition-counting functions $p_m(n)$ and $q_m(n)$.

Definition 2.1: Given a partition π , let $e(\pi)$ denote the number of even parts in π . Define $P_m(n)$ to be the set of partitions of n in which odd parts are distinct and all parts are less than or equal to $2m$, $Q_m(n)$ to be the set of partitions of n in which even parts are distinct and all parts are less than or equal to $2m-1$. We define

$$(9) \quad p_m(n) = \sum_{\pi \in P_m(n)} (-1)^{e(\pi)},$$

$$(10) \quad q_m(n) = \sum_{\pi \in Q_m(n)} (-1)^{e(\pi)},$$

so that

$$(11) \quad \frac{(-q; q^2)_m}{(-q^2; q^2)_m} = \sum_{n=0}^{\infty} p_m(n) q^n,$$

$$(12) \quad \frac{(q^2; q^2)_{m-1}}{(q; q^2)_m} = \sum_{n=0}^{\infty} q_m(n) q^n.$$

Theorem 2.1: If $r_3(n)$ is the number of representations of n as sum of three squares and if $p_m(n)$ and $q_m(n)$ are as defined by (9)-(10), then

$$(13) \quad r_3(n) = \sum_{m=1}^n \sum_{0 \leq i \leq (n-m)/2m} (-1)^i [2p_m(n-2im-m) + 4q_m(n-2im-m)].$$

Proof: Employing (11), (12), and the fact that

$$\frac{q^m}{1+q^{2m}} = \sum_{i=0}^{\infty} (-1)^i q^{2im+m}$$

in (8), we immediately have (13).

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NOTICE OF NOVEMBER 1992 VOLUME INDEX CORRECTION

- ◆ K. Atanassov's name was inadvertently omitted from the list of authors.
- ◆ K. Atanassov's coauthored article "Recurrent Formulas of the Generalized Fibonacci and Tribonacci Sequences" was incorrectly credited to Richard André-Jeannin.

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