SOME POLYNOMIAL IDENTITIES FOR THE FIBONACCI AND LUCAS NUMBERS

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It is well known that

(a) $F_{3n} = F_n\{5F_n^2 + 3(-1)^n\}$

Less well known are:

(b) $F_{5n} = F_n\{25F_n^4 + 25(-1)^n F_n^2 + 5\}$;

(c) $F_{7n} = F_n\{125F_n^6 + 175(-1)^n F_n^4 + 70F_n^2 + 7(-1)^n\}$.

In this paper we are concerned with proving a general formula which encompasses the above identities. That is, expresses F_{mn} as a polynomial in F_n for odd m. Also we prove two additional formulas which express F_{mn} / F_n as a polynomial in the Lucas numbers L_n. Our first theorem is

Theorem 1:

$$F_{(2q+1)n} = F_n \sum_{k=0}^{q} (-1)^{n(q+k)} \frac{2q+1}{q+k+1} 5^k \binom{q+k+1}{2k+1} F_n^{2k}, \quad n, q \geq 0.$$

Taking q = 1, 2, and 3, respectively in Theorem 1 gives us (a), (b), and (c) above. From Theorem 1 a couple of well-known results follow as corollaries.

Corollary 1.1: For $n \geq 0, p$ prime, we have

$$F_{pn} \equiv \left(\frac{5}{p}\right) F_n \pmod{p}.$$

Proof: Take $p = 2q+1, p$ prime, in Theorem 1, and by Euler's criterion, we have

$$5^{\frac{(p-1)}{2}} \equiv \left(\frac{5}{p}\right) \pmod{p}. \quad \square$$

Corollary 1.2: For prime p and q, we have $F_{pq} \equiv F_p F_q \pmod{pq}$.

Proof: From Corollary 1.1 with n = 1 and n =q, we have

$$F_p \equiv \left(\frac{5}{p}\right) \pmod{p} \text{ and } F_{pq} \equiv \left(\frac{5}{p}\right) F_q \pmod{p}, \text{ respectively.}$$

Hence, $F_{pq} \equiv F_p F_q \pmod{p}$.

Similarly, $F_{pq} \equiv F_p F_q \pmod{q}$. $\square$

Proof of Theorem 1: First we need two lemmas.

Lemma (i):

$$\left(x^{2m}+\frac{1}{x^{2m}}\right)+\left(x^{2m-2}+\frac{1}{x^{2m-2}}\right)+\cdots+\left(x^2+\frac{1}{x^2}\right)+1=\sum_{k=0}^{m}\frac{2m+1}{m+k+1}\binom{m+k+1}{2k+1}\left(x-\frac{1}{x}\right)^{2k}$$

Lemma (ii):

$$\left(x^{2m}+\frac{1}{x^{2m}}\right)-\left(x^{2m-2}+\frac{1}{x^{2m-2}}\right)+\cdots+(-1)^{m+1}\left(x^2+\frac{1}{x^2}\right)+(-1)^m$$
$$=\sum_{k=0}^{m}(-1)^{m+k}\frac{2m+1}{m+k+1}\binom{m+k+1}{2k+1}\left(x+\frac{1}{x}\right)^{2k}.$$

Now, from $F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$, where $\alpha+\beta=1$ and $\alpha\beta=-1$, we have for integer $p\ge 1$, $n\ge 1$,

$$\frac{F_{pn}}{F_n}=\frac{\alpha^{pn}-\beta^{pn}}{\alpha_n-\beta_n}=x^{p-1}+x^{p-2}y+x^{p-3}y^2+\cdots+xy^{p-2}+y^{p-1}, \tag{1.1}$$

where $x=\alpha^n$, $y=\beta^n=(-1)^n/x$.

Now, for odd p, the RHS of (1.1) is

$$\left(x^{p-1}+\frac{1}{x^{p-1}}\right)+(-1)^n\left(x^{p-3}+\frac{1}{x^{p-3}}\right)+\cdots+\left(x^2+\frac{1}{x^2}\right)+(-1)^n,\quad p\equiv 3 \pmod 4,$$

$$\left(x^{p-1}+\frac{1}{x^{p-1}}\right)+(-1)^n\left(x^{p-3}+\frac{1}{x^{p-3}}\right)+\cdots+(-1)^n\left(x^2+\frac{1}{x^2}\right)+1,\quad p\equiv 1 \pmod 4,$$

and $x+\frac{1}{x}=\alpha^n+\frac{1}{\alpha^n}=\alpha^n+(-1)^n\beta^n$. So that

$$x+\frac{1}{x}=(\alpha-\beta)F_n \quad \text{for odd } n. \tag{1.2}$$

$$x-\frac{1}{x}=(\alpha-\beta)F_n \quad \text{for even } n. \tag{1.3}$$

Since $\alpha-\beta=\sqrt{5}$, we have, from (1.2) and (1.3),

$$\left(x+\frac{1}{x}\right)^2=5F_n^2 \quad \text{for odd } n, \tag{1.4}$$

$$\left(x-\frac{1}{x}\right)^2=5F_n^2 \quad \text{for even } n. \tag{1.5}$$

So if we take $p = 2m+1$, and assume n is even, we have, from (1.1),

$$\frac{F_{pn}}{F_n} = \left(x^{2m} + \frac{1}{x^{2m}}\right) + \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + \left(x^2 + \frac{1}{x^2}\right) + 1.$$

Now apply Lemma (i) and use (1.5) to give Theorem 1 for even n. Similarly, setting $p = 2m+1$ and assuming n is odd, we have, from (1.1),

$$\frac{F_{pn}}{F_n} = \left(x^{2m} + \frac{1}{x^{2m}}\right) - \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + (-1)^{m+1}\left(x^2 + \frac{1}{x^2}\right) + (-1)^m.$$

Now apply Lemma (ii) and use (1.4) to give Theorem 1 for odd n. To complete the proof of Theorem 1, it only remains to prove Lemmas (i) and (ii). These can be proved by induction. For example, to prove Lemma (i), we set

$$P_m(x) = \left(x^{2m} + \frac{1}{x^{2m}}\right) + \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + \left(x^2 + \frac{1}{x^2}\right) + 1$$

and use

$$\left(x^2 + \frac{1}{x^2}\right)\left(x^{2m} + \frac{1}{x^{2m}}\right) = \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \left(x^{2m+2} + \frac{1}{x^{2m+2}}\right)$$

to give

$$\left(x^2 + \frac{1}{x^2}\right)P_m(x) = P_{m+1}(x) + P_{m-1}(x).$$

Hence,

(1.6) $$P_{m+1}(x) = \left\{\left(x - \frac{1}{x}\right)^2 + 2\right\}P_m(x) - P_{m-1}(x).$$

Then substitute the summation on the RHS of the identity in Lemma (i) for $P_m(x)$ and $P_{m-1}(x)$ in (1.6). Some careful work then gives $P_{m+1}(x)$ in the same form as the summation in Lemma (i). This proves Lemma (i). Lemma (ii) is proved in a similar manner, and this completes the proof of Theorem 1. □

For our next theorem we need some additional lemmas; these can be proved by induction in a way similar to that used to prove Lemma (i).

Lemma (iii):

$$\left(x^{2m} + \frac{1}{x^{2m}}\right) - \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + (-1)^{m+1}\left(x^2 + \frac{1}{x^2}\right) + (-1)^m = \sum_{k=0}^{m}\binom{m+k}{2k}\left(x - \frac{1}{x}\right)^{2k}.$$

Lemma (iv):

$$\left(x^{2m} + \frac{1}{x^{2m}}\right) + \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + \left(x^2 + \frac{1}{x^2}\right) + 1 = \sum_{k=0}^{m}(-1)^{m+k}\binom{m+k}{2k}\left(x + \frac{1}{x}\right)^{2k}.$$

Again from (1.1) with $p = 2m+1$, and noting that

$$x - \frac{1}{x} = \alpha^n + \beta^n = L_n \quad \text{for odd } n, \tag{1.7}$$

$$x + \frac{1}{x} = \alpha^n + \beta^n = L_n \quad \text{for even } n. \tag{1.8}$$

we have, from (1.7), (1.8), and Lemmas (iii) and (iv),

Theorem 2:

$$F_{(2q+1)n} = F_n \sum_{k=0}^{q} (-1)^{(n+1)(q+k)} \binom{q+k}{2k} L_n^{2k}, \quad n, q \geq 0.$$

A well-known formula follows as a corollary by taking $n = 1$; since $L_1 = 1$, we have

Corollary 2.1:

$$F_{2q+1} = \sum_{k=0}^{q} \binom{q+k}{2k}.$$

Our final theorem is similarly derived from the following two lemmas.

Lemma (v):

$$\left(x^{2m-1} - \frac{1}{x^{2m-1}}\right) - \left(x^{2m-3} - \frac{1}{x^{2m-3}}\right) + \cdots + (-1)^m \left(x^3 - \frac{1}{x^3}\right) + (-1)^{m-1} \left(x - \frac{1}{x}\right) = \sum_{k=1}^{m} \binom{m+k-1}{2k-1} \left(x - \frac{1}{x}\right)^{2k-1}.$$

Lemma (vi):

$$\left(x^{2m-1} + \frac{1}{x^{2m-1}}\right) + \left(x^{2m-3} + \frac{1}{x^{2m-3}}\right) + \cdots + \left(x^3 + \frac{1}{x^3}\right) + \left(x + \frac{1}{x}\right) = \sum_{k=1}^{m} (-1)^{m+k} \binom{m+k-1}{2k-1} \left(x + \frac{1}{x}\right)^{2k-1}.$$

Using Lemmas (v) and (vi) along with (1.1) gives

Theorem 3:

$$F_{2qn} = F_n \sum_{k=1}^{q} (-1)^{(n+1)(q+k)} \binom{q+k-1}{2k-1} L_n^{2k-1}, \quad n \geq 0, q \geq 1.$$

Again taking $n = 1$ gives us a well-known formula as a corollary.

Corollary 3.1:

$$F_{2q} = \sum_{k=1}^{q} \binom{q+k-1}{2k-1}.$$

The reader may notice that we appear to have one theorem missing. Namely, a theorem that expresses F_{2qn} as a polynomial F_n. However, to obtain such a formula we would need to be able to express the LHS of Lemma (v) exactly in powers of $\left(x + \frac{1}{x}\right)$ for odd n in (1.1), and the LHS of Lemma (vi) exactly in powers of $\left(x - \frac{1}{x}\right)$ for even n in (1.1), neither of which is possible.

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