SOME POLYNOMIAL IDENTITIES FOR THE FIBONACCI
AND LUCAS NUMBERS

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It is well known that
(a) \( F_{3n} = F_n \left(5F_n^2 + 3(-1)^n\right) \)

Less well known are:
(b) \( F_{5n} = F_n \left(25F_n^4 + 25(-1)^nF_n^2 + 5\right) \);
(c) \( F_{7n} = F_n \left(125F_n^6 + 175(-1)^nF_n^4 + 70F_n^2 + 7(-1)^n\right) \).

In this paper we are concerned with proving a general formula which encompasses the above identities. That is, expresses \( F_{mn} \) as a polynomial in \( F_n \) for odd \( m \). Also we prove two additional formulas which express \( F_{mn} / F_n \) as a polynomial in the Lucas numbers \( L_n \). Our first theorem is

**Theorem 1:**

\[
F_{(2q+1)n} = F_n \sum_{k=0}^{q} (-1)^n(q+k) \frac{2q+1}{q+k+1} 5^k \left(\frac{q+k+1}{2k+1}\right) F_n^{2k}, \quad n, q \geq 0.
\]

Taking \( q = 1, 2, \) and \( 3 \), respectively in Theorem 1 gives us (a), (b), and (c) above. From Theorem 1 a couple of well-known results follow as corollaries.

**Corollary 1.1:** For \( n \geq 0, p \) prime, we have

\[
F_{pn} \equiv \left(\frac{5}{p}\right) F_n \quad (\text{mod } p).
\]

**Proof:** Take \( p = 2q + 1, p \) prime, in Theorem 1, and by Euler's criterion, we have

\[
5^{(p-1)/2} \equiv \left(\frac{5}{p}\right) \quad (\text{mod } p). \quad \Box
\]

**Corollary 1.2:** For prime \( p \) and \( q \), we have \( F_{pq} \equiv F_p F_q \quad (\text{mod } pq) \).

**Proof:** From Corollary 1.1 with \( n = 1 \) and \( n = q \), we have

\[
F_p \equiv \left(\frac{5}{p}\right) \quad (\text{mod } p) \quad \text{and} \quad F_{pq} \equiv \left(\frac{5}{p}\right) F_q \quad (\text{mod } p), \quad \text{respectively.}
\]

Hence, \( F_{pq} \equiv F_p F_q \quad (\text{mod } p) \).

Similarly, \( F_{pq} \equiv F_p F_q \quad (\text{mod } q) \). \quad \Box
Proof of Theorem 1: First we need two lemmas.

Lemma (i):
\[
\left(x^{2m} + \frac{1}{x^{2m}} \right) + \left(x^{2m-2} + \frac{1}{x^{2m-2}} \right) + \cdots + \left(x^2 + \frac{1}{x^2} \right) + 1 = \sum_{k=0}^{m} \frac{2m+1}{m+k+1} \left(\frac{m+k+1}{2k+1} \right) \left(x - \frac{1}{x} \right)^{2k}
\]

Lemma (ii):
\[
\left(x^{2m} + \frac{1}{x^{2m}} \right) - \left(x^{2m-2} + \frac{1}{x^{2m-2}} \right) + \cdots + (-1)^{m+1} \left(x^2 + \frac{1}{x^2} \right) + (-1)^m = \sum_{k=0}^{m} (-1)^{m+k} \frac{2m+1}{m+k+1} \left(\frac{m+k+1}{2k+1} \right) \left(x + \frac{1}{x} \right)^{2k}.
\]

Now, from \(F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \), where \(\alpha + \beta = 1\) and \(\alpha \beta = -1\), we have for integer \(p \geq 1, n \geq 1,\)
\[
F_{np} = \frac{\alpha^{pn} - \beta^{pn}}{\alpha^n - \beta^n}\]
where \(x = \alpha^n, y = \beta^n = (-1)^n / x\).

Now, for odd \(p\), the RHS of (1.1) is
\[
\left(x^{p-1} + \frac{1}{x^{p-1}} \right) + (-1)^n \left(x^{p-3} + \frac{1}{x^{p-3}} \right) + \cdots + \left(x^2 + \frac{1}{x^2} \right) + (-1)^n, \quad p \equiv 3 \pmod{4},
\]
and \(x + \frac{1}{x} = \alpha^n + \frac{1}{\alpha^n} = \alpha^n + (-1)^n \beta^n\). So that
\[
x + \frac{1}{x} = (\alpha - \beta)F_n \quad \text{for odd } n.
\]

For even \(n\),
\[
x - \frac{1}{x} = (\alpha - \beta)F_n \quad \text{for even } n.
\]
Since \(\alpha - \beta = \sqrt{5}\), we have, from (1.2) and (1.3),
\[
\left(x + \frac{1}{x} \right)^2 = 5F_n^2 \quad \text{for odd } n,
\]
and
\[
\left(x - \frac{1}{x} \right)^2 = 5F_n^2 \quad \text{for even } n.
\]
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So if we take \( p = 2m + 1 \), and assume \( n \) is even, we have, from (1.1),

\[
\frac{F_{pn}}{F_n} = \left( x^{2m} + \frac{1}{x^{2m}} \right) + \left( x^{2m-2} + \frac{1}{x^{2m-2}} \right) + \cdots + \left( x^2 + \frac{1}{x^2} \right) + 1.
\]

Now apply Lemma (i) and use (1.5) to give Theorem 1 for even \( n \). Similarly, setting \( p = 2m + 1 \) and assuming \( n \) is odd, we have, from (1.1),

\[
\frac{F_{pn}}{F_n} = \left( x^{2m} + \frac{1}{x^{2m}} \right) - \left( x^{2m-2} + \frac{1}{x^{2m-2}} \right) + \cdots + (-1)^{m+1} \left( x^2 + \frac{1}{x^2} \right) + (-1)^m.
\]

Now apply Lemma (ii) and use (1.4) to give Theorem 1 for odd \( n \). To complete the proof of Theorem 1, it only remains to prove Lemmas (i) and (ii). These can be proved by induction. For example, to prove Lemma (i), we set

\[
P_m(x) = \left( x^{2m} + \frac{1}{x^{2m}} \right) + \left( x^{2m-2} + \frac{1}{x^{2m-2}} \right) + \cdots + \left( x^2 + \frac{1}{x^2} \right) + 1
\]

and use

\[
\left( x^2 + \frac{1}{x^2} \right)P_m(x) = P_{m+1}(x) + P_{m-1}(x).
\]

Hence,

\[
P_{m+1}(x) = \left( x - \frac{1}{x} \right)^2 P_m(x) - P_{m-1}(x).
\]

Then substitute the summation on the RHS of the identity in Lemma (i) for \( P_m(x) \) and \( P_{m-1}(x) \) in (1.6). Some careful work then gives \( P_{m+1}(x) \) in the same form as the summation in Lemma (i). This proves Lemma (i). Lemma (ii) is proved in a similar manner, and this completes the proof of Theorem 1.

For our next theorem we need some additional lemmas; these can be proved by induction in a way similar to that used to prove Lemma (i).

**Lemma (iii):**

\[
\left( x^{2m} + \frac{1}{x^{2m}} \right) - \left( x^{2m-2} + \frac{1}{x^{2m-2}} \right) + \cdots + (-1)^{m+1} \left( x^2 + \frac{1}{x^2} \right) + (-1)^m = \sum_{k=0}^{m} \left( \frac{m+k}{2k} \right) \left( x - \frac{1}{x} \right)^{2k}.
\]

**Lemma (iv):**

\[
\left( x^{2m} + \frac{1}{x^{2m}} \right) + \left( x^{2m-2} + \frac{1}{x^{2m-2}} \right) + \cdots + \left( x^2 + \frac{1}{x^2} \right) + 1 = \sum_{k=0}^{m} (-1)^{m+k} \left( \frac{m+k}{2k} \right) \left( x + \frac{1}{x} \right)^{2k}.
\]

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Again from (1.1) with \( p = 2m + 1 \), and noting that

\[
x - \frac{1}{x} = \alpha^n + \beta^n = L_n \quad \text{for odd } n,
\]

\[
x + \frac{1}{x} = \alpha^n + \beta^n = L_n \quad \text{for even } n.
\]

we have, from (1.7), (1.8), and Lemmas (iii) and (iv),

**Theorem 2:**

\[
F_{(2q+1)n} = F_n \sum_{k=0}^{q} (-1)^{(n+1)(q+k)} \binom{q+k}{2k} L_n^{2k}, \quad n, q \geq 0.
\]

A well-known formula follows as a corollary by taking \( n = 1 \); since \( L_1 = 1 \), we have

**Corollary 2.1:**

\[
F_{2q+1} = \sum_{k=0}^{q} \binom{q+k}{2k}.
\]

Our final theorem is similarly derived from the following two lemmas.

**Lemma (v):**

\[
\left( x^{2m-1} - \frac{1}{x^{2m-1}} \right) - \left( x^{2m-3} - \frac{1}{x^{2m-3}} \right) + \cdots + (-1)^m \left( x^3 - \frac{1}{x^3} \right) + (-1)^{m+1} \left( x - \frac{1}{x} \right) = \sum_{k=1}^{m} \binom{m+k-1}{2k-1} \left( x - \frac{1}{x} \right)^{2k-1}.
\]

**Lemma (vi):**

\[
\left( x^{2m-1} + \frac{1}{x^{2m-1}} \right) + \left( x^{2m-3} + \frac{1}{x^{2m-3}} \right) + \cdots + \left( x^3 + \frac{1}{x^3} \right) + \left( x + \frac{1}{x} \right) = \sum_{k=1}^{m} (-1)^{m-k} \binom{m+k-1}{2k-1} \left( x + \frac{1}{x} \right)^{2k-1}.
\]

Using Lemmas (v) and (vi) along with (1.1) gives

**Theorem 3:**

\[
F_{2q^n} = F_n \sum_{k=1}^{q} (-1)^{(n+1)(q+k)} \binom{q+k-1}{2k-1} L_n^{2k-1}, \quad n \geq 0, q \geq 1.
\]

Again taking \( n = 1 \) gives us a well-known formula as a corollary.

**Corollary 3.1:**

\[
F_{2q} = \sum_{k=1}^{q} \binom{q+k-1}{2k-1}.
\]

The reader may notice that we appear to have one theorem missing. Namely, a theorem that expresses \( F_{2q^n} \) as a polynomial \( F_n \). However, to obtain such a formula we would need to be able to express the LHS of Lemma (v) exactly in powers of \( x + \frac{1}{x} \) for odd \( n \) in (1.1), and the LHS of Lemma (vi) exactly in powers of \( x - \frac{1}{x} \) for even \( n \) in (1.1), neither of which is possible.

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