ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-478 ***Proposed by Gino Taddei, Rome, Italy***

Consider a string constituted by h labelled cells $c_1, c_2, \dots, c_h$. Fill these cells with the natural numbers 1, 2, ..., h according to the following rule: 1 in c_1, 2 in c_2, 3 in c_4, 4 in c_7, 5 in c_{11}, and so on. Obviously, whenever the subscript j of c_j exceeds h, it must be considered as reduced modulo h. In other words, the integer n $(1 \le n \le h)$ enters the cell $c_{j(n,h)}$, where

$$j(n,h) = \left\langle \frac{n^2 - n + 2}{2} \right\rangle_h,$$

and the symbol $\langle a \rangle_b$ denotes a if $a \le b$, and the remainder of a divided by b if $a > b$.

Determine the set of all values of h for which, at the end of the procedure, each cell has been entered by exactly one number.

H-479 ***Proposed by Richard André-Jeannin, Longwy, France***

Let $\{V_n\}$ be the sequence defined by

$$V_0 = 2,\ V_1 = P,\ \text{and}\ V_n = PV_{n-1} - QV_{n-2} \text{ for } n \ge 2,$$

where P and Q are real or complex parameters. Find a closed form for the sum

$$\sum_{k=1}^{n} \binom{2n-k-1}{n-1} P^k Q^{n-k} V_k.$$

H-480 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let p denote a prime $\equiv 1 \pmod{10}$.

(a) Prove that, for all $p \not\equiv 1 \pmod{1260}$, there exist positive integers k, u, and v such that

(i) $k | u^2$;

(ii) $p + 5k = (5u-1)(5v-1)$.

(b) Prove or disprove the conjecture that the restriction $p \not\equiv 1 \pmod{1260}$ in part (1) may be removed, i.e., part (a) is true for all $p \equiv 1 \pmod{10}$.

SOLUTIONS

Bunches of Recurrences

H-461 ***Proposed by Lawrence Somer, Washington, D.C.***
(Vol. 29, no. 4, November 1991)

Let $\{u_n\} = u(a, b)$ denote the Lucas sequence of the first kind satisfying the recursion relation $u_{n+2} = au_{n+1} + bu_n$, where a and b are nonzero integers and the initial terms are $u_0 = 0$ and $u_1 = 1$. The prime p is a primitive divisor of u_n if $p|u_n$ but $p\nmid u_m$ for $1 \le m \le n-1$. It is known (see [1], p. 200) for the Fibonacci sequence $\{F_n\} = u(1, 1)$ that, if p is an odd prime divisor of F_{2n+1}, where $n \ge 1$, then $p \equiv 1 \pmod 4$.

(i) Find an infinite number of recurrences $u(a, b)$ such that every odd primitive prime divisor p of any term of the form u_{2n+1} or u_{4n} satisfies $p \equiv 1 \pmod 4$, where $n \ge 1$.

(ii) Find an infinite number of recurrences $u(a, b)$ such that every odd primitive prime divisor p of any term of the form u_{4n} or u_{4n+2} satisfies $p \equiv 1 \pmod 4$, where $n \ge 1$.

Reference

1. E. Lucas. "Theorie des fonctions numeriques simplement périodiques." *Amer. J. Math.* **1** (1878):184-240, 289-321.

Solution by Paul S. Bruckman, Edmonds, WA

We write $P \in PD(u_n)$ if p is an odd primitive prime divisor of u_n. The following well-known result is stated in the form of a lemma.

Lemma: Suppose $m = x^2 + y^2$, where $x, y \in Z^+$. If p is any odd prime divisor of m, such that $p\nmid \gcd(x, y)$, then $p \equiv 1 \pmod 4$.

Next, we indicate some easily-derived results for a (generalized) Lucas sequence of the first kind:

$$u_n = \frac{r^n - s^n}{r - s}, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where

$$r = \tfrac{1}{2}(a + \theta), \quad s = \tfrac{1}{2}(a - \theta), \quad \theta = (a^2 + 4b)^{\frac{1}{2}}. \tag{2}$$

Note that

$$r + s = a, \quad r - s = \theta, \quad rs = -b. \tag{3}$$

Also, define the (generalized) Lucas sequence of the second kind as follows:

$$v_n = r^n + s^n, \quad n = 0, 1, 2, \ldots. \tag{4}$$

As we may readily verify:

$$u_{2n} = u_n v_n; \tag{5}$$

$$u_{2n+1} = bu_n^2 + u_{n+1}^2. \tag{6}$$

Also, it is clear that the u_n's and v_n's are integers for all n.

We will establish the following result, solving part (i) of the problem:

$$\text{If } a = i^2 - j^2,\ b = 1^2 j^2, \text{ where } i, j \in Z^+, \gcd(i, j) = 1, \text{ then } p \equiv 1 \pmod 4 \text{ for all prime } p \text{ such that } p \in PD(u_{2n+1}) \text{ or } p \in PD(u_{4n}),\ n \geq 1. \qquad (*)$$

Proof of (*): We note that $\theta^2 = a^2 + 4b = (i^2 - j^2)^2 + 4i^2 j^2 = (i^2 + j^2)^2$, so $\theta = i^2 + j^2$. Also, $r = i^2, s = -j^2$. We see from (6) that $u_{2n+1} = X^2 + Y^2$, where $X = iju_n, Y = u_{n+1}$. Also, from (4), $v_{2n} = X_1^2 + Y_1^2$, where $X_i = i^{2n}, Y_1 = j^{2n}$. If $p \in PD(u_{2n+1}), n \geq 1$, then $p | u_{2n+1}$, $p \nmid u_n$, $p \nmid u_{n+1}$. We cannot have $p | ij$, for otherwise, $p | X \Rightarrow p | Y = u_{n+1}$, a contradiction. Therefore, $p \nmid X, p \nmid Y$. Then, by the lemma, $p \equiv 1 \pmod 4$.

If $p \in PD(u_{4n}), n \geq 1$, then $p | u_{4n}, p \nmid u_{2n}$. Note that $u_{4n} = u_{2n} v_{2n}$ by (5). Thus, $p | v_{2n} = X_1^2 + Y_1^2$. Since $\gcd(i, j) = 1$, also $\gcd(X_1, Y_1) = 1$. By the Lemma, $p \equiv 1 \pmod 4$. This completes the proof of $(*)$.

Also, we shall prove the following result, which solves part (ii):

$$\text{If } a = i^2 + j^2, b = -i^2 j^2, \text{ where } i, j \in Z^+, \gcd(i, j) = 1, i > j, \text{ then } p \equiv 1 \pmod 4 \text{ for all prime } p \text{ such that } p \in PD(u_{4n}) \text{ or } p \in PD(u_{4n+2}), n \geq 1. \qquad (**)$$

Proof of (**): We note that $\theta^2 = a^2 + 4b = (i^2 + j^2)^2 - 4i^2 j^2 = (i^2 - j^2)^2$, so $\theta = i^2 - j^2$. Also, $r = i^2, s = j^2$, and so $v_n = X_2^2 + Y_2^2$, where $X_2 = i^n, Y_2 = j^n$. If $p \in PD(u_{2n})$, $n \geq 1$, then $p | u_{2n}, p \nmid u_n$. Using (5), $p | v_n = X_2^2 + Y_2^2$. Since $\gcd(i, j) = 1$ also $\gcd(X_2, Y_2) = 1$. By the Lemma, $p \equiv 1 \pmod 4$. Since $2n = 4n'$ or $4n' + 2$, we see that $(**)$ is proven.

In summary, we that i and j in $(*)$ and $(**)$ are arbitrary natural numbers, subject only to the condition that $\gcd(i, j) = 1$ [and $i > j$ in $(**)$]. Hence, there are infinitely many sequences $u(a, b)$, with a and b as given in $(*)$ and $(**)$, that provide solutions to the two parts of the problem.

Also solved by the proposer.

Root of the Problem

H-462 ***Proposed by Ioan Sadoveaanuv, Ellensburg, WA (Vol. 30, no. 1, February 1992)***

Let $G(x) = x^k + a_1 x^{k-1} + \cdots + a_k$ be a polynomial with c a root of order p. If $G^{(p)}(x)$ denotes the p^{th} derivative of $G(x)$, show that $\{n^p c^{n-p} / G^{(p)}(c)\}$ is a solution of the recurrence $u_n = c^{n-k} - a_1 u_{n-1} - a_2 u_{n-2} - \cdots - a_k u_{n-k}$.

Solution by C. Georghiou, University of Patras, Patras, Greece

We will use the operator method of Difference Calculus (see, e.g., Marray R. Spiegel, *Calculus of Finite Differences and Difference Equations* [New York: McGraw-Hill, 1971], p. 156). Let $G(x) = (x-c)^p g(x)$. Then $g(c) = G^{(p)}(c) / p!$ $(\neq 0)$. The given recurrence is written as $G(E) u_n = c^n$, where E is the shift operator, i.e., $Eu_n = u_{n+1}$. Therefore, the solution is

$$u_n = \frac{1}{G(E)} c^n = \frac{1}{(E-c)^p g(E)} c^n = \frac{1}{(E-c)^p} \frac{c^n}{g(c)} = \frac{p!}{G^{(p)}(c)} c^n \frac{1}{(cE-c)^p} 1 = \frac{p! c^{n-p}}{G^{(p)}(c)} \frac{1}{\Delta^p} 1.$$

Now, from the Summation Calculus, we have

$$\Delta^{-p}1=\frac{n^{(p)}}{p!}+\sum_{k=1}^{p}A_k\frac{n^{(p-k)}}{(p-k)!} \tag{1}$$

where, as usual, $n^{(k)}=n(n-1)\dots(n-k+1)$ is the factorial function, and $A_1, A_2, \dots, A_k$ are arbitrary constants. But it is known that

$$n^p=n^{(p)}+\sum_{k=0}^{p-1}S_p^{(k)}n^{(k)} \tag{2}$$

where $S_p^{(k)}$ are the Stirling Numbers of the Second Kind. If we choose $A_{p-k}=k!S_p^{(k)}/p!$ then (1), in view of (2), becomes $\Delta^{-p}1=n^p/p!$ and the assertion follows readily.

Also solved by P. Bruckman and F. Flanigan.

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H-463 ***Proposed by Paul S. Bruckman, Edmonds, WA***
(Vol. 30, no. 1, February 1992)

Establish the identity: $$\sum_{n=1}^{\infty}\Phi(n)\left(\frac{z^n}{1-z^{2n}}\right)=\frac{z(1+z+z^2)}{(1-z^2)^2}, \tag{1}$$

where $z\in C, |z|<1$, and Φ is the Euler totient function. As special cases of (1), obtain the following identities:

$$\sum_{n=1}^{\infty}\Phi(2n)/F_{2ns}=\sqrt{5}/L_s^2,\ s=1,3,5,\dots; \tag{2}$$

$$\sum_{n=1}^{\infty}\Phi(2n-1)/L_{(2n-1)s}=F_s\sqrt{5}/L_s^2,\ s=1,3,5,\dots; \tag{3}$$

$$\sum_{n=1}^{\infty}\Phi(n)/F_{ns}=(L_s+1)/F_s^2\sqrt{5},\ s=2,4,6,\dots; \tag{4}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\Phi(n)/F_{ns}=(L_s-1)/F_s^2\sqrt{5},\ s=2,4,6,\dots; \tag{5}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\Phi(2n)/F_{2ns}=\begin{cases}1/F_s^2\sqrt{5}, & s=1,3,5,\dots;\\ \sqrt{5}/L_s^2, & s=2,4,6,\dots;\end{cases} \tag{6}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\Phi(2n-1)/F_{(2n-1)s}=L_s/F_s^2\sqrt{5},\ s=1,3,5,\dots; \tag{7}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\Phi(2n-1)/L_{(2n-1)s}=F_s\sqrt{5}/L_s^2,\ s=2,4,6,\dots. \tag{8}$$

Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

For $|z|<1$,

$$\sum_{n=1}^{\infty}\Phi(n)\frac{z^n}{1-z^{2n}}=\sum_{n=1}^{\infty}\sum_{q\text{ odd}}\Phi(n)z^{qn}.$$

For odd t and $s \geq 0$, the coefficient of z^k, where $k = 2^s t$, is

$$\sum_{d|t} \Phi(2^s d) = \Phi(2^s) \sum_{d|t} \Phi(d) = \Phi(2^s) \cdot t = \begin{cases} 2^{s-1} t & \text{if } s > 0, \\ t & \text{if } s = 0. \end{cases}$$

Therefore,

$$\sum_{n \text{ odd}} \Phi(n) \frac{z^n}{1 - z^{2n}} = \sum_{n=1}^{\infty} (2n+1) z^{2n+1} = \frac{z(1+z^2)}{(1-z^2)^2} \tag{$*$}$$

and

$$\sum_{n \text{ even}} \Phi(n) \frac{z^n}{1 - z^{2n}} = \sum_{n=1}^{\infty} n z^{2n} = \frac{z^2}{(1-z^2)^2}, \tag{$**$}$$

which prove (1). Letting $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, we have $\alpha\beta = -1$ and the identities

$$\frac{1}{\sqrt{5}} \frac{1}{F_{ns}} = \frac{1}{\alpha^{ns} - \beta^{ns}} = \frac{\beta^{ns}}{(-1)^{ns} - \beta^{2ns}}. \tag{A}$$

$$\frac{1}{L_{ns}} = \frac{1}{\alpha^{ns} + \beta^{ns}} = \frac{\beta^{ns}}{(-1)^{ns} + \beta^{2ns}}. \tag{B}$$

$$\frac{\beta^{2s}}{(1-\beta^{2s})^2} = \frac{(\alpha\beta)^{2s}}{[\alpha^s - (\alpha\beta)^s \beta^s]^2} = \begin{cases} 1/L_s^2 & \text{if } s \text{ is odd} \\ 1/5F_s^2 & \text{if } s \text{ is even.} \end{cases} \tag{C}$$

$$\frac{\beta^s(1+\beta^{2s})}{(1-\beta^{2s})^2} = \frac{(\alpha\beta)^s[\alpha^s + (\alpha\beta)^s \beta^s]}{[\alpha^s - (\alpha\beta)^s \beta^s]^2} = \begin{cases} -F_s\sqrt{5}/L_s^2 & \text{if } s \text{ is odd,} \\ L_s/5F_s^2 & \text{if } s \text{ is even.} \end{cases} \tag{D}$$

$$\frac{\beta^{2s}}{(1+\beta^{2s})^2} = \frac{(\alpha\beta)^{2s}}{[\alpha^s + (\alpha\beta)^s \beta^s]^2} = \begin{cases} 1/5F_s^2 & \text{if } s \text{ is odd,} \\ 1/L_s^2 & \text{if } s \text{ is even.} \end{cases} \tag{E}$$

$$\frac{\beta^s(1-\beta^{2s})}{(1+\beta^{2s})^2} = \frac{(\alpha\beta)^s[\alpha^s - (\alpha\beta)^s \beta^s]}{[\alpha^s + (\alpha\beta)^s \beta^s]^2} = \begin{cases} -L_s/5F_s^2 & \text{if } s \text{ is odd,} \\ F_s\sqrt{5}/L_s^2 & \text{if } s \text{ is even.} \end{cases} \tag{F}$$

To prove (2)-(8), proceed as follows:

(2) For odd s, it follows from (A), ($**$), and (C) that

$$\frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{\Phi(2n)}{F_{2ns}} = \sum_{n \text{ even}} \Phi(n) \frac{\beta^{ns}}{1-\beta^{2ns}} = \frac{\beta^{2s}}{(1-\beta^{2s})^2} = \frac{1}{L_s^2}.$$

(3) For even s, it follows from (B), ($*$), and (D) that

$$\sum_{n=1}^{\infty} \frac{\Phi(2n-1)}{L_{(2n-1)s}} = -\sum_{n \text{ odd}} \Phi(n) \frac{\beta^{ns}}{1-\beta^{2ns}} = -\frac{\beta^s(1+\beta^s)}{(1-\beta^{2s})^2} = \frac{F_s\sqrt{5}}{L_s^2}.$$

(4) For even s, it follows from (A), (1), (C), and (D) that

$$\frac{1}{\sqrt{5}}\sum_{n=1}^{\infty}\frac{\Phi(n)}{F_{ns}}=\sum_{n=1}^{\infty}\Phi(n)\frac{\beta^{ns}}{1-\beta^{2ns}}=\frac{\beta^{s}(1+\beta^{2s})+\beta^{2s}}{(1-\beta^{2s})^2}=\frac{L_s+1}{5F_s^2}.$$

(5) For even s, it follows from (A), (1), (C), and (D) that

$$\frac{1}{\sqrt{5}}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Phi(n)}{F_{ns}}=-\sum_{n=1}^{\infty}\Phi(n)\frac{(-\beta^{s})^n}{1-(-\beta^{s})^{2n}}=\frac{\beta^{s}(1+\beta^{2s})-\beta^{2s}}{(1-\beta^{2s})^2}=\frac{L_s-1}{5F_s^2}.$$

(6) It follows from (A), (∗∗), and (E) that

$$\frac{1}{\sqrt{5}}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Phi(2n)}{F_{2ns}}=-\sum_{n\text{ even}}\Phi(n)\frac{(i\beta^{s})^n}{1-(i\beta^{s})^{2n}}=\frac{\beta^{2s}}{(1+\beta^{2s})^2}=\begin{cases}1/5F_s^2 & \text{if } s \text{ is odd,}\\ 1/L_s^2 & \text{if } s \text{ is even.}\end{cases}$$

(7) For odd s, it follows from (A), (∗), and (F) that

$$\frac{1}{\sqrt{5}}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Phi(2n-1)}{F_{(2n-1)s}}=-\frac{1}{i}\sum_{n\text{ odd}}\Phi(n)\frac{(i\beta^{s})^n}{1-(i\beta^{s})^{2n}}=-\frac{\beta^{s}(1-\beta^{2s})}{(1+\beta^{2s})^2}=\frac{L_s}{5F_s^2}.$$

(8) For even s, it follows from (A), (∗), and (F) that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Phi(2n-1)}{L_{(2n-1)s}}=\frac{1}{i}\sum_{n\text{ odd}}\Phi(n)\frac{(i\beta^{s})^n}{1-(i\beta^{s})^{2n}}=\frac{\beta^{s}(1-\beta^{2s})}{(1+\beta^{2s})^2}=\frac{F_s\sqrt{5}}{L_s^2}.$$

Also solved by C. Georghiou, P. Haukkanen, R. Hendel, and the proposer.

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