FIBONACCI TRANSMISSION LINES

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1. INTRODUCTION

The Fibonacci sequence applies to many diverse areas in science and technology [1, 2]. In a book review by Brother Alfred Brousseau, a significant observation was made which will surely be true for all time: "Enter the magic door which leads to the wonderful world of Fibonacci" [3]. The author found the magic door and is overwhelmed at the beauty of the landscape. This paper will present those findings that helped the author locate the "magic door" and to be fascinated by what is inside. Many other investigators have significantly helped light the way for these findings [5, 6, 7, 8].

2. PRELIMINARIES

In Figure 1(a), a two wire transmission line having a characteristic impedance of Z_0 ohms is shown. The input terminals are marked $a-b$. If a resistive load whose value is chosen equal to the characteristic impedance is placed at a quarter or odd quarter wavelength from the input terminals, the input impedance will be equal to the characteristic impedance and will result in a "matched" line condition. In fact, as long as the load is matched to the characteristic impedance of the line, it can be placed anywhere along the line without changing the input impedance. From an ideal point of view, this is a desired condition; but, it is not achieved in practice whenever two or more loads are connected to the line. At this point, it will be practically advantageous to normalize all connected loads to the characteristic impedance of the line. All connected loads equal to Z_0 will have the normalized value of 1 or unity and will be referred to as unit loads. If an actual load value is needed, the normalized value can be multiplied by Z_0 ohms. The next step, as well as succeeding steps, will be to periodically "load" the line at quarter wavelength $\lambda/4$ or odd quarter wavelength intervals with unit loads and to determine for each load the resultant input impedance. Figure 1(b) shows two loads connected across the line. The second load and associated quarter wavelength section of line places in parallel with the first unit load another unit load which when combined on a parallel resistor basis results in an equivalent load of one half unit. This equivalent load at the input terminals produces a value of two unit loads because of the inversion properties of a quarter wavelength section of line. The two unit loads and the input value of two are coincidental. If a third unit load is connected across the line at a quarter wavelength from load 2, a total of three loads are connected and the length of the line is three quarter wavelength long relative to the input terminals $a-b$. This is shown schematically in Figure 1(c). Since the previous results showed that the input impedance is equivalent to two unit loads, this places two unit loads in parallel with one unit load which results in an equivalent load of $2/3$. At the input terminals, the input impedance becomes $3/2$. If this process is continued for "n" sections, it is found that the normalized input impedance of a periodically loaded transmission line is equal to the ratio of two Fibonacci numbers, namely, F_{n+1}/F_n. This is shown in Figure 1(d). Such a line will be referred to as a "Fibonacci Transmission Line." And if this line is extended in the limit to large values of "n," the normalized input impedance is found to be the "golden ratio," namely, 1.618... [8].

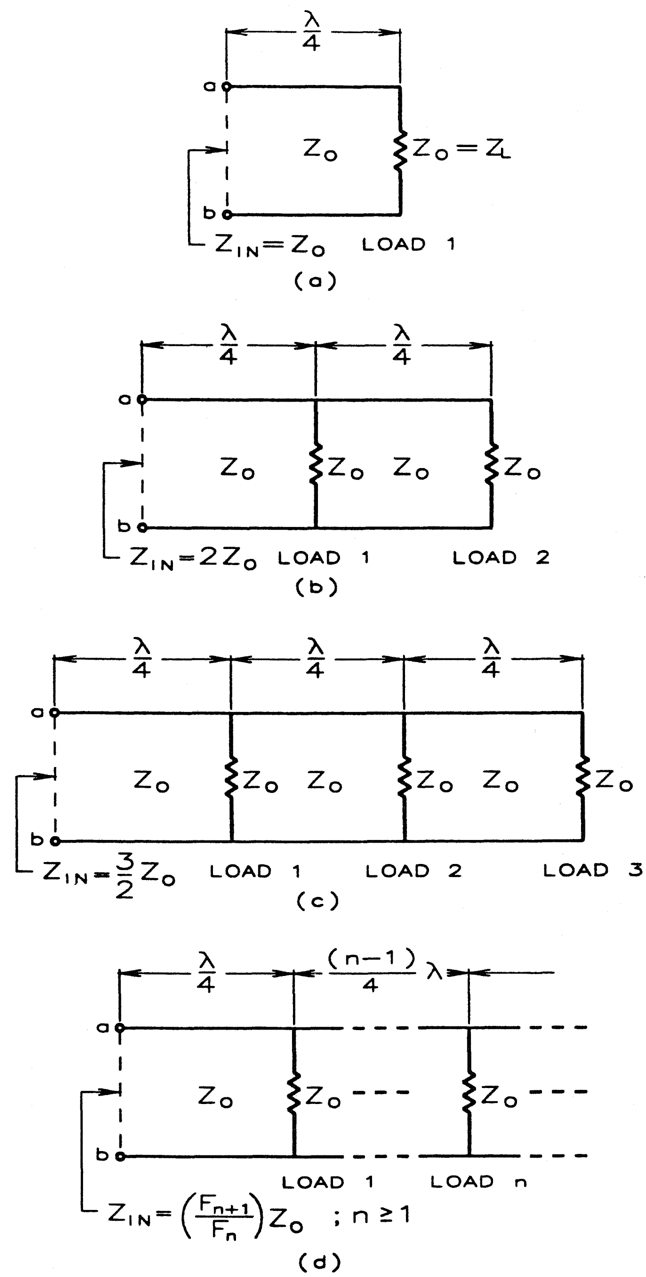


FIGURE 1
Periodically Loaded Transmission Line

3. ANALYSIS

There are many important parameters associated with every transmission line. Some of these parameters, such as the normalized input impedance, reflection coefficient, along with voltage and power ratios are shown in Table 1. Importantly, all parameters are functions of the Fibonacci numbers and related functions.

TABLE 1

Fibonacci Transmission Line Parameters

n	1	2	3	4	5	n	$n\to\infty$
$\frac{Z_{IN}}{Z_0}$	1	2	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{8}{5}$	$\frac{F_{n+1}}{F_n}$	1.6180...
Γ	0	$-\frac{1}{3}$	$-\frac{1}{5}$	$-\frac{2}{8}$	$-\frac{3}{13}$	$-\frac{F_{n-1}}{F_{n+2}}$	−0.2360...
$VSWR$	1	2	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{8}{5}$	$\frac{F_{n+1}}{F_n}$	1.6180...
$\frac{V_{IN}}{V_s}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{8}{13}$	$\frac{F_{n+1}}{F_{n+2}}$	0.6180...
$\frac{P_{IN}}{P_A}$	1	$4\cdot\frac{2}{9}$	$4\cdot\frac{6}{25}$	$4\cdot\frac{15}{64}$	$4\cdot\frac{40}{169}$	$4\cdot\frac{F_{n+1}F_n}{(F_{n+2})^2}$	0.9442...

The input impedance for a section of lossless transmission line is given by (see [9]),

$$Z_{1N} = \frac{Z_0(Z_L + jZ_0 \tan b1)}{(Z_0 + jZ_L \tan b1)}, \tag{1}$$

where

Z_0 = characteristic impedance
Z_L = load impedance
b = phase constant = $2\Pi/\lambda$
λ = wavelength = v/f
v = wave velocity along the line
f = frequency of voltage and current waves on the line
l = length along the line
$j = \sqrt{-1}$.

For the special case when the line is equal to a quarter wavelength, equation (1) becomes

$$Z_{IN} = \frac{(Z_0)^2}{Z_L}. \tag{2}$$

In Figure 1(b), the input impedance of the second load and line is Z_0 ohms. This equivalent impedance when combined with load #1 becomes

$$Z_L = \frac{Z_0 Z_0}{(Z_0 + Z_0)} = \frac{Z_0}{2}. \tag{3}$$

When the value of Z_L is used in equation (2), the input impedance becomes

$$Z_{IN} = 2Z_0. \tag{4}$$

If another cell, as shown in Figure 1(c), is connected to the first two cells, the equivalent load can be determined by combining the resistive loads,

$$Z_L = \frac{2Z_0 Z_0}{(2Z_0 + Z_0)} = \frac{2Z_0}{3}, \tag{5}$$

using equation (2), the input impedance becomes

$$Z_{IN} = \left(\frac{3}{2}\right) Z_0. \tag{6}$$

For Figure 1(d), the input impedance for n sections is given by

$$Z_{IN} = \left(\frac{F_{n+1}}{F_n}\right) Z_0; \; n \geq 1, \tag{7}$$

where F_{n+1} and F_n are two consecutive Fibonacci numbers. For large n, equation (7) becomes

$$Z_{IN} = \lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n}\right) Z_0 = (1.61803\ldots) Z_0. \tag{8}$$

The reflection coefficient and voltage standing wave ratio are important parameters that describe the behavior of transmission lines in relation to another connected transmission line or to a connected load. The reflection coefficient is defined as the ratio of reflected to incident voltage or current wave amplitudes. In general, it will be a complex quantity having amplitude and angle values. In terms of a connected load, it is determined by

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}. \tag{9}$$

For a Fibonacci transmission line, (9) becomes

$$\Gamma = -\frac{F_{n-1}}{F_{n+2}}; \; n \geq 1. \tag{10}$$

The voltage standing wave ratio is determined by the ratio of maximum to minimum voltage amplitudes along the line. In terms of the reflection coefficient,

$$VSWR = \frac{1+|\Gamma|}{1-|\Gamma|}. \tag{11}$$

Using (10), the $VSWR$ is

$$VSWR = \frac{F_{n+1}}{F_n}; \; n \geq 1. \tag{12}$$

The circuit shown in Figure 2 will be used to determine the input voltage to a Fibonacci transmission line. The generator impedance is made equal to Z_0 for convenience. By using the voltage divider rule, the input voltage can be written as

$$V_{IN} = \frac{Z_{IN}V_s}{Z_{IN}+Z_0} \tag{13}$$

$$\frac{V_{IN}}{V_s} = \frac{F_{n+1}}{F_{n+2}};\ n \geq 1. \tag{14}$$

The last parameter considered is the ratio of input power, P_{IN}, to available power, P_A:

$$P_{IN} = \frac{V_{IN}^2}{Z_{IN}}; \tag{15}$$

$$P_A = \frac{V_s^2}{4Z_0}; \tag{16}$$

$$\frac{P_{IN}}{P_A} = \frac{4F_{n+1}F_n}{(F_{n+2})^2} = \frac{(F_{n+2})^2-(F_{n-1})^2}{(F_{n+2})^2} = 1-\left(\frac{F_{n-1}}{F_{n+2}}\right)^2 = 1-\Gamma^2. \tag{17}$$

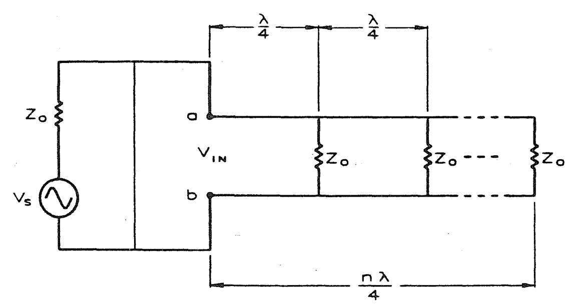


FIGURE 2

Fibonacci Transmission Line Circuit

It is interesting to consider *what if* situations for Fibonacci Transmission Lines (FTL) and ladder-type electrical networks. First, for an n loaded FTL: What is the resultant input impedance of an FTL if each of the n loads of Z_0 ohms is replaced by another FTL which has $m - Z_0$ ohm loaded sections? A schematic of the *what if* FTL is shown in Figure 3.

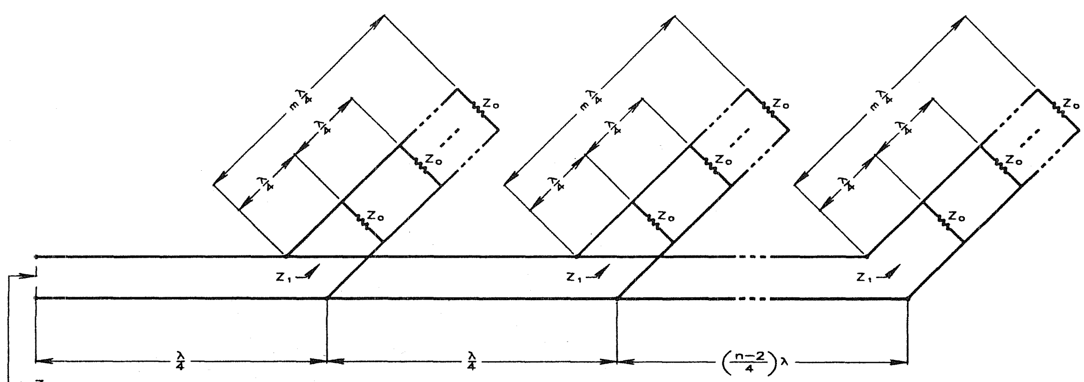


FIGURE 3

Fibonacci Transmission Line of Fibonacci Transmission Lines

From basic transmission line theory, if an impedance Z_1 is connected as a load in a $\lambda/4$ section of line having a characteristic impedance of Z_0, as shown in Figure 4(a), the input impedance is

$$Z_{IN_1} = \frac{Z_0^2}{Z_1}. \tag{18}$$

If another identical load, Z_1, is connected, as shown in Figure 4(b), the input impedance is

$$Z_{IN_2} = \frac{Z_1^2 + Z_0^2}{Z_1}. \tag{19}$$

If a third load, Z_1, is connected, as shown in Figure 4(c), the input impedance is

$$Z_{IN_3} = \frac{Z_0^2\left(2Z_1^2 + Z_0^2\right)}{\left(Z_1^2 + Z_0^2\right)Z_1}. \tag{20}$$

If a fourth load, Z_1, is connected, as shown in Figure 4(d), is

$$Z_{IN_4} = \frac{Z_1^4 + 3Z_1^2Z_0^2 + Z_0^4}{\left(2Z_1^2 + Z_0^2\right)Z_1}. \tag{21}$$

If a fifth load, Z_1, is connected, as shown in Figure 4(e), the input impedance is

$$Z_{IN_5} = \frac{Z_0^2\left(3Z_1^4 + 4Z_1^2Z_0^2 + Z_0^4\right)}{\left(Z_1^4 + 3Z_1^2Z_0^2 + Z_0^4\right)Z_1}. \tag{22}$$

Let a_1 be the parameter for the normalized Z_1 impedance, $a_1 = \frac{Z_1}{Z_0} = \frac{F_{m+1}}{F_m}$, then:

$$Z_{IN_1} = \frac{Z_0^2}{Z_1} = \frac{Z_0}{\frac{Z_1}{Z_0}} = \frac{Z_0}{a_1}; \tag{23}$$

$$Z_{IN_2} = \frac{Z_1^2 + Z_0^2}{Z_1} = \frac{Z_0}{a_1}\left(1 + a_1^2\right); \tag{24}$$

$$Z_{IN_3} = \frac{Z_0^2}{Z_1}\left(\frac{2Z_1^2 + Z_0^2}{Z_1^2 + Z_0^2}\right) = \frac{Z_0}{a_1}\left(\frac{1 + 2a_1^2}{1 + a_1^2}\right); \tag{25}$$

$$Z_{IN_4} = \frac{Z_0}{a_1} \cdot \frac{\left(1 + 3a_1^2 + a_1^2\right)}{\left(1 + 2a_1^2\right)}; \tag{26}$$

$$Z_{IN_5} = \frac{Z_0}{a_1} \cdot \frac{\left(1 + 4a_1^2 + 3a_1^4\right)}{\left(1 + 3a_1^2 + a_1^4\right)}. \tag{27}$$

The polynomials in the numerator and denominator are Jacobsthal polynomials (see [10, 11]).

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x) \tag{28}$$

with $J_1(x) = J_2(x) = 1$.

In the Fibonacci transmission line structure,

$$x = a_1^2. \tag{29}$$

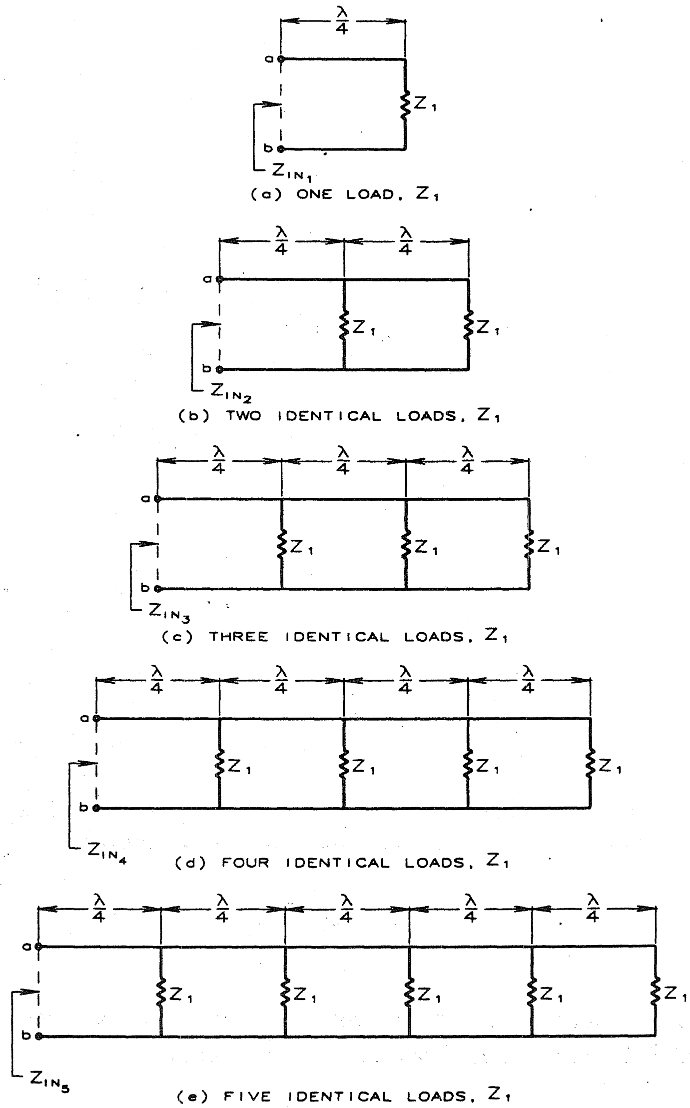


FIGURE 4
Periodically Loaded Transmission Lines

Using the Jacobsthal polynomials, the input impedances can be rewritten as:

$$Z_{IN_1} = \frac{Z_0}{a_1} \cdot \frac{J_2}{J_1}; \tag{30}$$

$$Z_{IN_2} = \frac{Z_0}{a_1}\left(1 + a_1^2\right) = \frac{Z_0}{a_1} \cdot \frac{J_3}{J_2}; \tag{31}$$

$$Z_{IN_3} = \frac{Z_0}{a_1}\left(\frac{1 + 2a_1^2}{1 + a_1^2}\right) = \frac{Z_0}{a_1} \cdot \frac{J_4}{J_3}. \tag{32}$$

Finally, for n connected Z_1 loads,

$$Z_{IN_n} = \frac{Z_0}{a_1} \cdot \frac{J_{n+1}(a_1)}{J_n(a_1)}. \tag{33}$$

The general term of the Jacobsthal polynomials is given by

$$J_n(a_1) = \frac{1}{\sqrt{1 + 4a_1^2}}\left[\left(\frac{1 + \sqrt{1 + 4a_1^2}}{2}\right)^n - \left(\frac{1 - \sqrt{1 + 4a_1^2}}{2}\right)^n\right]. \tag{34}$$

For the case $a_1 = 1$, the Jacobsthal sequence is the Fibonacci sequence. Other expressions for Jacobsthal polynomials are:

$$J_n = \frac{2\sqrt{a_1^{2n}}}{\sqrt{1 + 4a_1^2}} \cdot \sinh\left\{n\left[\ln\left(\frac{1 + \sqrt{1 + 4a_1^2}}{2\sqrt{a_1^2}}\right)\right]\right\}; \tag{35}$$

$$J_n = \frac{2\sqrt{a_1^{2n}}}{\sqrt{1 + 4a_1^2}} \cdot \cosh\left\{n\left[\ln\left(\frac{1 + \sqrt{1 + 4a_1^2}}{2\sqrt{a_1^2}}\right)\right]\right\}. \tag{36}$$

If each matched load in a Fibonacci transmission line is replaced by another Fibonacci transmission line, as shown in Figure 3, the resultant input impedance is given by

$$Z_{IN}^{(1)} = Z_0\left(\frac{F_m}{F_{m+1}}\right)\left(\frac{J_{n+1}}{J_n}\right), \tag{37}$$

where the superscript number in parentheses represents the first replacement of each connected load by an m-loaded FTL. In the special case $m = n$, $Z_{IN}^{(1)}$ becomes

$$Z_{IN}^{(1)} = Z_0\left(\frac{F_n}{F_{n+1}}\right)\left(\frac{J_{n+1}(a_1)}{J_n(a_1)}\right). \tag{38}$$

If a second replacement of each Z_0 in the first replacement transmission line is made, the input impedance becomes

$$Z_{IN}^{(2)} = Z_0\left(\frac{F_{n+1}}{F_n}\right)\left[\frac{J_n(a_1)}{J_{n+1}(a_1)}\right]\left[\frac{J_{n+1}(a_2)}{J_n(a_2)}\right], \tag{39}$$

where

$$a_1 = \frac{F_{n+1}}{F_n} \text{ and } a_2 = \frac{F_n}{F_{n+1}}\left[\frac{J_{n+1}(a_1)}{J_n(a_1)}\right].$$

If a third replacement is made, the input impedance becomes

$$Z_{IN}^{(3)} = Z_0\left(\frac{F_n}{F_{n+1}}\right)\left[\frac{J_{n+1}(a_1)}{J_n(a_1)}\right]\left[\frac{J_n(a_2)}{J_{n+1}(a_2)}\right]\left[\frac{J_{n+1}(a_3)}{J_n(a_3)}\right], \tag{40}$$

where

$$a_3 = \frac{F_{n+1}}{F_n}\left[\frac{J_n(a_1)}{J_{n+1}(a_1)}\right]\left[\frac{J_{n+1}(a_2)}{J_n(a_2)}\right].$$

Next, for ladder electrical networks, the input resistance for m half-T sections is given by equation (9) in reference [8]. Rewriting the reference equation,

$$Z_{IN} = \left(\frac{F_{2m+1}}{F_{2m}}\right)R;\ m \geq 1, \tag{41}$$

where R is the value in ohms of each resistor in the ladder network. The ladder network is shown schematically in Figure 5(a). Like the FTL, if each resistor R in the ladder is replaced by n half-T sections in a ladder configuration with individual input impedance of

$$Z_{IN} = \left(\frac{F_{2n+1}}{F_{2n}}\right)R;\ n \geq 1, \tag{42}$$

the resultant input impedance of a ladder of ladders is

$$Z_{IN} = \left(\frac{F_{2m+1}}{F_{2m}}\right)\left(\frac{F_{2n+1}}{F_{2n}}\right)R;\ m \geq 1 \text{ and } n \geq 1. \tag{43}$$

For the special case $m = n$, the input resistance is

$$Z_{IN} = \left(\frac{F_{2n+1}}{F_{2n}}\right)^2 R;\ n \geq 1. \tag{44}$$

An implementation of a ladder of ladders is shown in Figure 5(b).

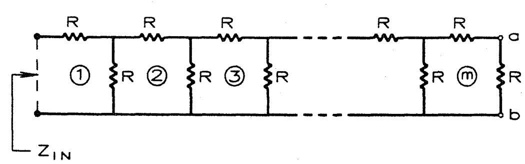
To conclude this development, the FTL and ladder will be extended to include K and M replacements or iterations of the basic symmetrical networks, respectively. After K replacements, the input impedance, $Z_{IN}^{(K)}$, can be written as

$$Z_{IN}^{(K)} = Z_0\left(\frac{F_{n+1}}{F_n}\right)^{(-1)^K} \prod_{i=1}^{K}\left[\frac{J_{n+1}(a_i)}{J_n(a_i)}\right]^{(-1)^{K-1}} ; \; K \geq 1, \tag{45}$$

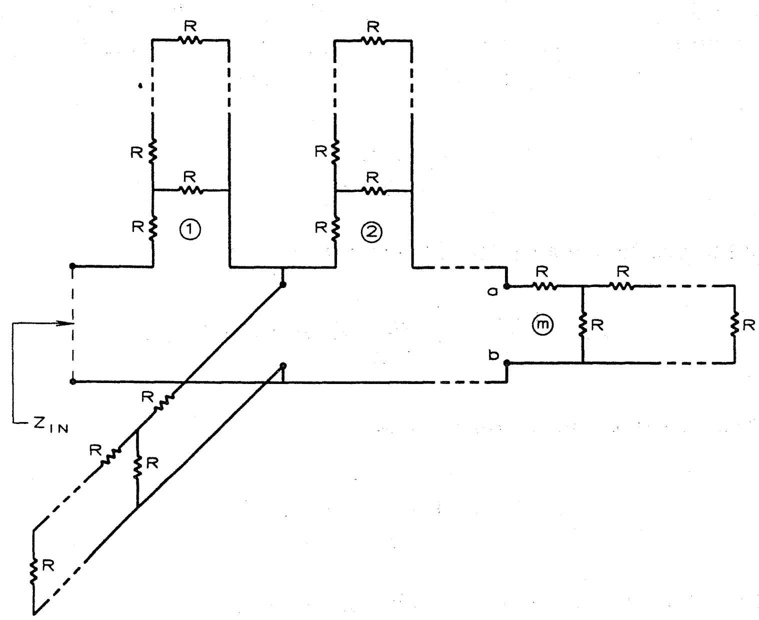
and, for the symmetrical ladder network, the input impedance is

$$Z_{IN} = \left(\frac{F_{2n+1}}{F_{2n}}\right)^{M+1} R; \; n \geq 1 \text{ and } M = 0, \pm 1, \pm 2, \ldots. \tag{46}$$

A brief look inside the M door shows that the input impedance ranges from an open circuit to a short circuit as M increases positively or negatively, respectively. Interestingly, for ladder networks, the equivalent resistance of each element increases for positive M and decreases for negative M. This suggests series paths for positive M and parallel paths for negative M. Figure 6 shows a ladder of ladders for different values of M.



(a) m HALF–T SECTION LADDER NETWORK



(b) m HALF–T SECTION LADDER OF LADDERS

FIGURE 5

Ladder of Ladders Network

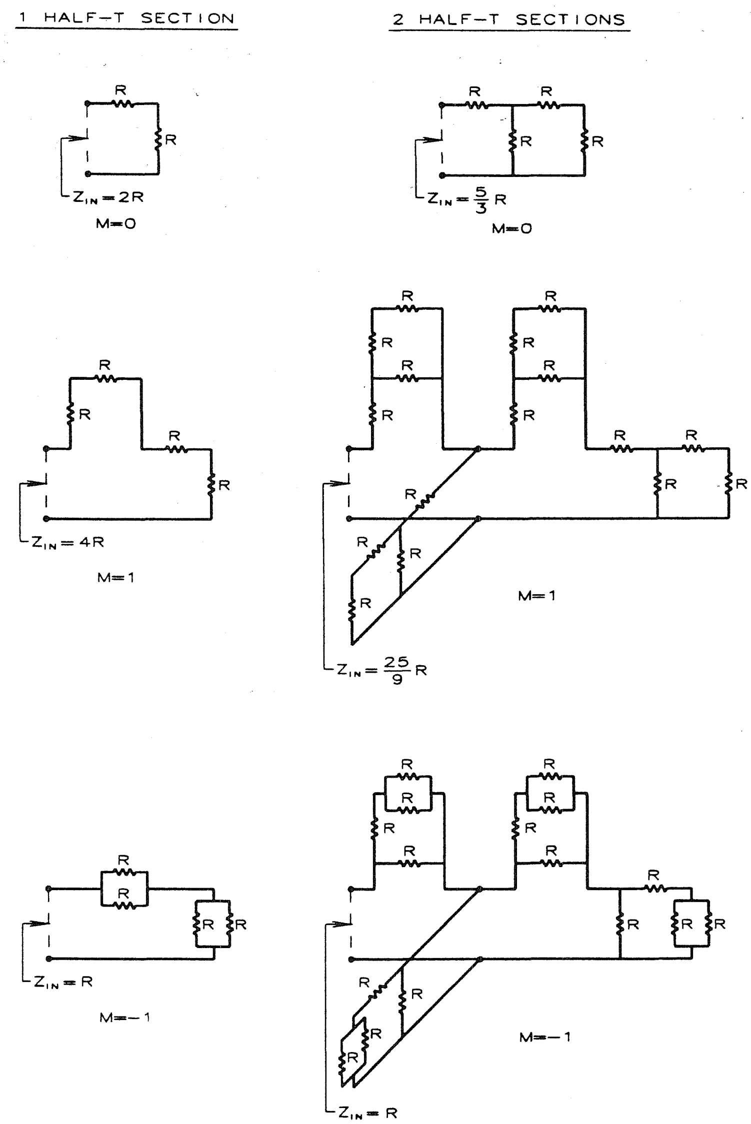


FIGURE 6
Ladder Iterations

4. CONCLUSIONS

The results of this investigation shows that the Fibonacci sequence and related functions can be used to analyze periodically loaded wave transmission structures. This is an important result that opens new doors to a variety of transmission systems investigations. For example, these results can be used to analyze local area networks (LAN) that use transmission lines to tie computers togather or for array-type antennas excited by transmission lines and used for either reception or transmission. Another important finding of this investigation is the extension of

Fibonacci transmission lines and ladder networks to higher-order structures by an iterative process. Importantly, the results presented in this paper open many new doors which lead to new doors and more doors or doors {doors[doors(doors)]...}. In conclusion, the world of Fibonacci provides many opportunities for new and exciting discoveries.

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