RATIONAL CHEBYSHEV APPROXIMATIONS OF ANALYTIC FUNCTIONS

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1. RATIONAL CHEBYSHEV APPROXIMATIONS OF ANALYTIC FUNCTIONS

We proceed to establish the main result of this paper: a general procedure to obtain rational Chevyshev approximations of analytic functions. Let $f(z)$ be analytic at z_0. Then, by composition, $g(z) = f(\cos z + z_0 - 1)$ is analytic at the origin. Hence, we can write

$$g(z) = f(\cos z + z_0 - 1) = \sum_{n=0}^{\infty} g^{(2n)}(0)\frac{z^{2n}}{(2n)!}. \tag{1.1}$$

If an explicit expansion of $f(\cos z + z_0 - 1)$ is not available, then successive coefficients in (1.1) are found directly from the formula for Maclaurin expansions, i.e., by simply calculating successive derivatives of (1.1) and setting $z = 0$. To wit,

$$g(0) = f(z_0), \tag{1.2}$$

$$g''(0) = -f'(z_0), \tag{1.3}$$

$$g^{(\text{iv})}(0) = 3f''(z_0) + f'(z_0), \tag{1.4}$$

$$g^{(\text{vi})}(0) = -15f'''(z_0) - 15f''(z_0) - f'(z_0), \tag{1.5}$$

$$g^{(\text{viii})}(0) = 105f^{(\text{iv})}(z_0) + 210f'''(z_0) + 63f''(z_0) + f'(z_0), \tag{1.6}$$

$$g^{(\text{x})}(0) = -945f^{(\text{v})}(z_0) - 3150f^{(\text{iv})}(z_0) - 2205f'''(z_0) - 255f''(z_0) - f'(z_0), \tag{1.7}$$

$$\begin{aligned} g^{(\text{xii})}(0) = {} & 10395f^{(\text{vi})}(z_0) + 51975f^{(\text{v})}(z_0) + 65835f^{(\text{iv})}(z_0) + 21120f'''(z_0) \\ & + 1023f''(z_0) + f'(z_0), \end{aligned} \tag{1.8}$$

$$\begin{aligned} g^{(\text{xiv})}(0) = {} & -135135f^{(\text{vii})}(z_0) - 945945f^{(\text{vi})}(z_0) - 1891890f^{(\text{v})}(z_0) - 1201200f^{(\text{iv})}(z_0) \\ & - 195195f'''(z_0) - 4095f''(z_0) - f'(z_0), \end{aligned} \tag{1.9}$$

etc.; the derivatives of odd order at the origin being at zero, since $g(z)$ is an even function of z.

Now, consider the expression

$$\begin{aligned} g(z) \approx {} & A_1 \cos z - A_2 g(z) \cos z + A_3 \cos 2z - A_4 g(z) \cos 2z + \cdots \\ & + A_{2s-1} \cos sz - A_{2s} g(z) \cos sz, \end{aligned} \tag{1.10}$$

where the A_k's are constants to be determined, and the $\approx$ in (1.10) is to be interpreted in the sense that the Maclaurin expansions of both sides agree through the first $2s$ terms.

Note that both sides of (1.10) are, of course, even, as they should be.

Observe that the Cauchy product of $g(z)$ and $\cos mz$ is

$$g(z)\cos mz = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{g^{(2n-2k)}(0)(-1)^k m^{2k} z^{2n}}{(2n-2k)!\,(2k)!}. \tag{1.11}$$

Since $\cos mz$ is entire, the above Cauchy product will have the same circle of convergence that equation (1.1) has (see [4]).

Using (1.11) to equate powers of z in (1.10) we find, after multiplying through by $(-1)^n(2n)!$,

$$\begin{aligned}(-1)^n g^{(2n)}(0) = A_1 &- A_2\sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{2k}g^{(2n-2k)}(0) + 2^{2n}A_3\\ &- A_4\sum_{k=0}^{n}(-1)^{n-k}2^{2k}\binom{2n}{2k}g^{(2n-2k)}(0)+\cdots+s^{2n}A_{2s-1}\\ &- A_{2s}\sum_{k=0}^{n}(-1)^{n-k}s^{2k}\binom{2n}{2k}g^{(2n-2k)}(0),\end{aligned} \tag{1.12}$$

where $\binom{n}{k}$ is the binomial coefficient.

Letting $n=0,1,2,\ldots,2s-1$ in (1.12), we find an algebraic system of $2s$ equations with $2s$ unknowns for the determination of the A's. Then, $g(z)$ is found as

$$g(z)\approx\frac{A_1\cos z + A_3\cos 2z+\cdots+A_{2s-1}\cos sz}{1+A_2\cos z+A_4\cos 2z+\cdots+A_{2s}\cos sz}. \tag{1.13}$$

Now, in equation (1.13), replace the above z by $\cos^{-1}(z-z_0+1)$, and make use of the defining equation for Chebyshev polynomials of the first kind $T_n(z)=\cos(n\cos^{-1}z)$, recalling the relation between $f(z)$ and $g(z)$ to obtain

$$f(z)\approx\frac{A_1T_1(z-z_0+1)+A_3T_2(z-z_0+1)+\cdots+A_{2s-1}T_s(z-z_0+1)}{T_0(z-z_0+1)+A_2T_1(z-z_0+1)+\cdots+A_{2s}T_s(z-z_0+1)}, \tag{1.14}$$

which gives a rational Chebyshev approximation of $f(z)$ where the only restriction which has been assumed is analyticity of the function at z_0.

Power series of the form given in (1.1) are sometimes found Taylor-made in the literature. For instance, see [6],

$$\exp(\cos z-1)=1-\frac{1}{2}z^2+\frac{1}{6}z^4-\frac{31}{720}z^6+\cdots, \tag{1.15}$$

where the general coefficient is

$$\frac{(-1)^n 2^{1-n}}{n!\,(2n)!}\sum_{k=0}^{n-1}(-1)^k 2^k(-n)_k\sum_{r=0}^{n-k-1}\frac{(2k-2n)_r}{r!}(n-k-r)_{2n}, \tag{1.16}$$

where $(\alpha)_n=\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$, $(\alpha)_0=1$, $\alpha\neq 0$, is Pochhammer's symbol. In series (1.15), $z_0=0$.

Also, see [5],

$$\log \cos z = \sum_{n=1}^{\infty} (-1)^n (2^{2n}-1) 2^{2n-1} B_{2n} z^{2n} / [n(2n)!], \tag{1.17}$$

where the B_{2n} are Bernoulli numbers (see [1]). In the series (1.17), $z_0 = 1$.

It will be noticed that the coefficient of $f^{(j)}(z_0)$ in the sum for $g^{(2i)}(0)$, $(i = 1, 2, \ldots, 2s-1,$ $j = 1, 2, \ldots, 2s-1)$, exemplified in the list given at the beginning of this section, equations (1.2) through (1.9), is also the coefficient of $\cos jz$, evaluated at $z = 0$, in

$$\frac{d^{2j}}{dz^{2j}}(\exp(\cos z - 1)).$$

This provides a simple computer algorithm for generating these coefficients. This observation is due to one of the authors (Rosenthal).

2. ADAPTING THE ALGORITHM FOR THE GENERALIZED HYPERGEOMETRIC FUNCTION

The method we have developed enables us to find, in simple fashion, a rational Chebyshev approximation for the generalized hypergeometric function ${}_pF_q(z)$:

$${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = 1 + \sum_{n=1}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n z^n}{(b_1)_n (b_2)_n \cdots (b_q)_n n!}, \tag{2.1}$$

where none of the b's is zero or a negative integer (see [14]).

The derivative of (2.1) is given by (see [14])

$$\frac{a_1 a_2 \cdots a_p}{b_1 b_2 \cdots b_q}\, {}_pF_q(a_1+1, a_2+1, \ldots, a_p+1; b_1+1, b_2+1, \ldots, b_q+1; z). \tag{2.2}$$

The value of the hypergeometric function at the origin is 1. Hence, choosing $z_0 = 0$, it is quite simple to determine successive derivatives of the ${}_pF_q(z)$ at the origin to find, with the aid of equations (1.2) through (1.9), the values of $g(z)$ and its derivatives at $z = 0$.

Note that $g(0)$ and its derivatives at the origin will be given as rational functions of the coefficients of the ${}_pF_q(z)$. In particular, if these coefficients are themselves rational, then the rational Chebyshev approximation will involve only rational coefficients.

As the reader no doubt knows, many known functions are special cases (at most with a multiplicative monomial) of the generalized hypergeometric function. We will choose Bessel functions,

$$J_n(z) = \frac{(z/2)^n}{\Gamma(1+n)}\, {}_0F_1\left(-; 1+n; -\tfrac{1}{4}z^2\right), \tag{2.3}$$

to illustrate the algorithm.

It will be recalled that we mentioned, following (2.2), that, if the parameters appearing in the hypergeometric function are rational numbers, then the A's, the solutions of the system of equations (1.12), are also rational numbers. This holds true in most of the important cases. For this reason, we found it desirable to make use of a program (we chose REDUCE [15]) that did not execute the operation of division, so that the A's would be given in fractional form.

We close this section by making a comment that is probably obvious to the reader. If one wishes to go from a given s, the highest order of the Chebyshev polynomials in (1.14), to $s+1$ in the system of equations (1.12), then the matrix of the coefficients for $s+1$ will be the same as that for s, except that two rows and two columns will be added. Hence, knowing the inverse of the $2s \times 2s$ matrix one can find the inverse of the $(2s+2)\times(2s+2)$ matrix by using the method of partitioning in the technique known as "inversion by bordering."

3. ILLUSTRATING THE ALGORITHM

We will now give some examples of rational Chebyshev approximations obtained by use of the procedure outlined in the previous section. To list the approximations, we will give them in the following format:

$$f(z) \approx \frac{az^k(p_0z^n + p_1z^{n-1} + p_2z^{n-2} + \cdots + p_{n-1}z + p_n)}{b(q_0z^m + q_1z^{m-1} + q_2z^{m-2} + \cdots + q_{m-1}z + q_m)}, \tag{3.1}$$

where $k+n \le s$, and $m \le s$. For each s we will simply list the coefficients in (3.1).

$$\underline{\underline{f(z) = J_0(z)}}$$

$$\underline{s=2}$$

$$a=4, k=0, n=2, p_0=2, p_1=0, p_2=-3;$$
$$b=1, m=2, q_0=5, q_1=0, q_2=-12.$$

$$\underline{s=3}$$

$$a=-1, k=0, n=3, p_0=69, p_1=51, p_2=-368, p_3=-272;$$
$$b=1, m=3, q_0=23, q_1=17, q_2=368, q_3=272.$$

$$\underline{s=6}$$

$$a=12, k=0, n=6, p_0=57\,76742, p_1=0, p_2=-1838\,79735,$$
$$p_3=0, p_4=10070\,89152, p_5=0, p_6=-7895\,61600;$$

$$b=1, m=6, q_0=60\,35647, q_1=0, q_2=3705\,82236, q_3=0,$$
$$q_4=97163\,85024, q_5=0, q_6=-94747\,39200.$$

The reader should observe that the magnitude of the coefficients increases quite rapidly with increasing s. We shall shortly see that the quality of the approximation also improves very rapidly as s increases.

$$\underline{s=10}$$

$$a=300, k=0, n=10,$$

$$\begin{aligned}
p_0 &= 2114\,63570\,00545\,36614, & p_1 &= 0,\\
p_2 &= -4\,28033\,75450\,19518\,86781, & p_3 &= 0,\\
p_4 &= 281\,17868\,03665\,81890\,18624, & p_5 &= 0,\\
p_6 &= -6194\,13498\,85928\,62663\,77984, & p_7 &= 0,\\
p_8 &= 31326\,83622\,73236\,69829\,38624, & p_9 &= 0,\\
p_{10} &= -23739\,05902\,96182\,29215\,88736;
\end{aligned}$$

$$b = 1,\ m = 10,$$

$$\begin{aligned}
q_0 &= 3272\,56614\,14968\,07057, & q_1 &= 0,\\
q_2 &= 9\,84654\,95148\,64179\,66500, & q_3 &= 0,\\
q_4 &= 1597\,67150\,04330\,42594\,24000, & q_5 &= 0,\\
q_6 &= 1\,57441\,70286\,74100\,89721\,60000, & q_7 &= 0,\\
q_8 &= 76\,17621\,44097\,57337\,57624\,32000, & q_9 &= 0,\\
q_{10} &= -71\,21717\,70888\,54687\,64766\,20800.
\end{aligned}$$

4. NUMERICAL VALUES AND GRAPHS OF SOME RATIONAL CHEBYSHEV APPROXIMATIONS

In this section we present the results of evaluating the rational forms given in section 3. The runs for different values of the parameter s will be contrasted with the tabulated values given in [1]. The latter will be taken, for purposes of comparison, as exact values.

$$f(z) = J_0(z)$$

Exact values		**$s = 2$**		**$s = 3$**	
z	$J_0(z)$	z	$J_0(z)$	z	$J_0(z)$
0.0	1.00000 00000 00000	0.0	1.00000 00000 00000	0.0	1.00000 00000 00000
0.1	0.99750 15620 66040	0.1	0.99748 95397 48954	0.1	0.99750 15615 24048
0.2	0.99002 49722 39576	0.2	0.98983 05084 74576	0.2	0.99002 49376 55860
0.3	0.97762 62465 38296	0.3	0.97662 33766 23377	0.3	0.97762 58545 68055
0.4	0.96039 82266 59563	0.4	0.95714 28571 42857	0.4	0.96039 60396 03960
0.5	0.93846 98072 40813	0.5	0.93023 25581 39535	0.5	0.93846 15384 61539
0.6	0.91200 48634 97211	0.6	0.89411 76470 58824	0.6	0.91198 04400 97800
0.7	0.88120 08886 07405	0.7	0.84607 32984 29319	0.7	0.88114 00848 99939
0.8	0.84628 73527 50480	0.8	0.78181 81818 18182	0.8	0.84615 38461 53846
0.9	0.80752 37981 22545	0.9	0.69433 96226 41509	0.9	0.80725 75847 70970
1.0	0.76519 76865 57967	1.0	0.57142 85714 28571	1.0	0.76470 58823 52941
1.1	0.71962 20185 27511	1.1	0.38991 59663 86554	1.1	0.71876 81580 47647
1.2	0.67113 27442 64363	1.2	0.09999 99999 99999	1.2	0.66972 47706 42202
1.3	0.62008 59895 61509	1.3	-0.42816 90140 84507	1.3	0.61786 31995 47767
1.4	0.56685 51203 74289	1.4	-1.67272 72727 27273	1.4	0.56347 43875 27840
1.5	0.51182 76717 35918	1.5	-8.00000 00000 00000	1.5	0.50684 93150 68493
1.6	0.45540 21676 39381	1.6	10.60000 00000 00000	1.6	0.44827 58620 68966
1.7	0.39798 48594 46109	1.7	4.53877 55102 04082	1.7	0.38803 59978 82478
1.8	0.33998 64110 42558	1.8	3.31428 57142 85714	1.8	0.32640 33264 03326
1.9	0.28181 85593 74385	1.9	2.79008 26446 28099	1.9	0.26364 09994 90056
2.0	0.22389 07791 41236	2.0	2.50000 00000 00000	2.0	0.20000 00000 00000
2.1	0.16660 69803 31990	2.1	2.31641 79104 47761	2.1	0.13571 77853 99314
2.2	0.11036 22669 22174	2.2	2.19016 39344 26230	2.2	0.07101 72744 72169
2.3	0.05553 97844 45602	2.3	2.09826 98961 93771	2.3	0.00610 61531 23532
2.4	0.00250 76832 97244	2.4	2.02857 14285 71429	2.4	-0.05882 35294 11765
2.5	-0.04838 37764 68198	2.5	1.97402 59740 25974	2.5	-0.12359 55056 17978

	$s = 2$	$s = 3$
Error at z = 0.1:	1.20E-05	5.42E-10
Error at z = 1.5:	8.512	0.005
Error at z = 2.5:	-2.022	0.075

$s = 6$		$s = 10$	
z	$J_0(z)$	z	$J_0(z)$
0.0	1.00000 00000 00000	0.0	1.00000 00000 00000
0.1	0.99750 15620 66040	0.1	0.99750 15620 66040
0.2	0.99002 49722 39576	0.2	0.99002 49722 39576
0.3	0.97762 62465 38249	0.3	0.97762 62465 38296
0.4	0.96039 82266 57938	0.4	0.96039 82266 59564
0.5	0.93846 98072 14225	0.5	0.93846 98072 40813
0.6	0.91200 48632 17224	0.6	0.91200 48634 97211
0.7	0.88120 08863 32675	0.7	0.88120 08886 07405
0.8	0.84628 73359 87120	0.8	0.84628 73527 50480
0.9	0.80752 36414 25774	0.9	0.80752 37981 22545
1.0	0.76519 88991 81058	1.0	0.76519 76865 57967
1.1	0.71962 28437 77746	1.1	0.71962 20185 27512
1.2	0.67113 39900 87712	1.2	0.67113 27442 64364
1.3	0.62008 81243 85951	1.3	0.62008 59895 61514
1.4	0.56685 88740 92599	1.4	0.56685 51203 74305
1.5	0.51183 42373 87263	1.5	0.51182 76717 35967
1.6	0.45541 34601 77972	1.6	0.45540 21676 39523
1.7	0.39800 38749 36571	1.7	0.39798 48594 46502
1.8	0.34001 77192 20127	1.8	0.33998 64110 43589
1.9	0.28186 89650 63377	1.9	0.28181 85593 76972
2.0	0.22397 01919 55021	2.0	0.22389 07791 47447
2.1	0.16672 95358 25093	2.1	0.16660 69803 46316
2.2	0.11054 77454 23837	2.2	0.11036 22669 54003
2.3	0.05581 53758 52507	2.3	0.05553 97845 13916
2.4	0.00291 01468 75270	2.4	0.00250 76834 39234
2.5	-0.04780 55089 54713	2.5	-0.04838 37761 81732

Error at z = 0.1: 0	Error at z = 0.1: 0
Error at z = 1.5:-6.57E-06	Error at z = 1.5:-4.90E-14
Error at z = 2.5:-5.78E-04	Error at z = 2.5:-2.86E-10

The algorithm is seen to be very stable. As the value of s increases, the quality of the approximations improves notably. The last example above, $J_0(z)$ for $s = 10$, gives remarkable agreement throughout the range $0 \le |z| \le 2.5$.

5. ZEROS OF THE DENOMINATOR POLYNOMIALS OF THE RATIONAL CHEBYSHEV APPROXIMATIONS

If in equation (1.10) we let s increase without bound, then both sides will represent the same function since their Maclaurin expansions agree for all terms. In this case, equation (1.13) will have an infinite series in both the numerator and denominator. The values of z for which the series in the denominator converges to zero will be singular points of $g(z)$, unless the series in the numerator also converges to zero there. As equation (1.13) stands, it being an approximate relation, it is conceivable that the right-hand side may have poles which are not singular points of the function $g(z)$. This implies, of course, that the right-hand side of equation (1.14) may also have poles which are not singular points of $f(z)$. These would be the so-called *spurious poles*. Let us look at this phenomenon somewhat more closely for the example given in Section 3.

The denominator polynomial of the rational Chebyshev approximation for the Bessel function $J_0(z)$ corresponding to $s = 10$ has real zeros at the points

$$z = \pm 0.95778\ 12766\ 24968\ 22726\ 05909\ 45945.$$

Yet, the graph given in Figure 1, and the table of values of this function do not seem to indicate any abnormal behavior in the neighborhood of this point. However, if we analyze the rational approximation within ±E-18 of this point, then the rational form is seen to undergo marked oscillations with nearly infinite slope. Nevertheless, as soon as we are within ±E-17 of the point in question, the erratic behavior disappears and the algorithm again represents the correct values of the Bessel function $J_0(z)$.

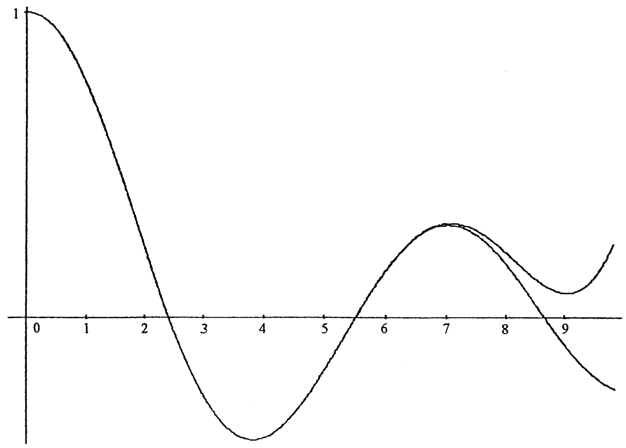

Figure 1

This figure shows the Bessel function of the first kind of order zero, $J_0(z)$ plotted against the rational Chebyshev approximation corresponding to $s = 10$. After $z = 9$, the Bessel function continues to oscillate, while the approximation separates from this behavior. The two functions move apart after $z = 7$. The algorithm approximates the first zero of the Bessel function to be 2.40482 55580, and the second zero to be 5.51960 87207. These results compare favorably with the correct values 2.40482 55577 and 5.52007 81103 given in [1].

We shall now speak of the significance of these roots. The highly localized character of the oscillation indicates that the numerator polynomial also has zeros which are very close to the zeros of the denominator polynomial. This is indeed the case for all of the examples we studied. The numerator polynomial of the $s = 10$ approximation of the Bessel function, for instance, has real zeros at the points

$$z = \pm 0.95778\ 12766\ 24968\ 22150\ 32913\ 84229$$

which match the zeros of the denominator polynomial through seventeen decimal places. The oscillatory behavior is then simply a reflection of the computer's arithmetic inability to handle 0/0. The algorithm, we see, is a self-correcting one that introduces zeros in the numerator and denominator polynomials in a way that ensures the correct approximation to the function for a given value of s.

In essence, our method provides a rational approximation $P_s(z)/Q_s(z)$ such that its Taylor expansion about the point z_0 agrees with the Taylor expansion of $f(z)$ through the first $2s$ terms. This requirement may be written as

$$Q_s(z)f(z) - P_s(z) = (z - z_0)^{2s+1} \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

and it is equivalent to the criterion for choosing the s^{th} diagonal entry in the Padé table for $z_0 = 0$.

Because of the proximity of the real zeros of the numerator and denominator polynomials of the Bessel function approximation corresponding to $s = 10$, we chose to divide out the zeros and try out the outcome against the tabulated values given before. The resulting expression is:

$$a = 300, k = 0, n = 8$$

$$\begin{array}{ll}
p_0 = 2114\,63570\,00545\,36614.00000\,00000\,0, & p_1 = 0, \\
p_2 = -4\,26093\,90407\,09759\,89175.46176\,79548\,8, & p_3 = 0, \\
p_4 = 277\,26992\,93536\,91260\,65928.27801\,88600\,0, & p_5 = 0, \\
p_6 = -5939\,78281\,24995\,79471\,89316.44693\,04600\,0, & p_7 = 0, \\
p_8 = 25878\,00631\,84966\,42221\,73861.61878\,68000\,0, &
\end{array}$$

$$b = 1, m = 8$$

$$\begin{array}{ll}
q_0 = 3272\,56614\,14968\,07057.00000\,00000\,0, & q_1 = 0, \\
q_2 = 9\,87657\,02358\,79227\,26257.68351\,76325\,9, & q_3 = 0, \\
q_4 = 1606\,73172\,24978\,36037\,41999.24516\,09500\,0, & q_5 = 0, \\
q_6 = 1\,58915\,63013\,73742\,21409\,56470.02415\,10000\,0, & q_7 = 0, \\
q_8 = 77\,63401\,89554\,89925\,73188\,73022.34814\,00000\,0 &
\end{array}$$

The tabulated values resulting from this approximation are:

z	$J_0(z)$
0.0	1.00000 00000 00000
0.1	0.99750 15620 66040
0.2	0.99002 49722 39576
0.3	0.97762 62465 38296
0.4	0.96039 82266 59564
0.5	0.93846 98072 40813
0.6	0.91200 48634 97211
0.7	0.88120 08886 07405
0.8	0.84628 73527 50480
0.9	0.80752 37981 22545
1.0	0.76519 76865 57967
1.1	0.71962 20185 27512
1.2	0.67113 27442 64364
1.3	0.62008 59895 61514
1.4	0.56685 51203 74305
1.5	0.51182 76717 35967
1.6	0.45540 21676 39523
1.7	0.39798 48594 46502
1.8	0.33998 64110 43589
1.9	0.28181 85593 76972
2.0	0.22389 07791 47447
2.1	0.16660 69803 46316
2.2	0.11036 22669 54003
2.3	0.05553 97845 13916
2.4	0.00250 76834 39234
2.5	-0.04838 37761 81732

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Error at z = 0.1: 0
Error at z = 1.5:-4.90E-14
Error at z = 2.5:-2.86E-10
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These are exactly the same values, to fifteen-decimal accuracy, obtained with the $s = 10$ approximation of the Bessel function $J_0(z)$ before the roots are divided out!—These results imply a substantial saving in computer time since the number of divisions required for a given approximation is reduced by two.

A comment is in order, though it is probably obvious to the reader. The results shown in the above table were obtained by dividing the numerator polynomial by its real roots, and the denominator polynomial by its corresponding real roots. Slightly better accuracy is obtained (though the above table does not indicate it) if we divide both numerator and denominator polynomials by either the real roots of the numerator *or* the real roots of the denominator since, in this case, all we are doing is dividing numerator and denominator of the $s = 10$ approximation by a common factor.

It is worth emphasizing that the rational Chebyshev approximations our algorithm provides are not optimal, in the sense that error does not remain constant within the range of approximation. Rather, error is least when one is sufficiently near the point z_0 and the quality of the approximation deteriorates as we move away from the point in question. The importance of the method lies, we believe, in the extreme simplicity with which it can provide rational Chebyshev approximations of any accuracy for a wide variety of functions. These nonoptimal approximations may easily be used to obtain optimal Chebyshev approximations. Several algorithms have been developed to this effect.

Let us speak now of the origin of the problem that has occupied us in the last five sections.

6. SOME HISTORY

About one hundred and twenty-five years ago, the Russian mathematician Pafnuty Lvovich Chebyshev (1821-1894) set himself the problem of finding the best rational approximation of a continuous function specified on an interval $[a, b]$. Specifically, he wanted to determine parameters $p_0, p_1, \ldots, p_n$; $q_0, q_1, \ldots, q_m$ in the expression

$$Q(x) = s(x)\frac{p_0x^n + p_1x^{n-1} + \cdots + p_n}{q_0x^m + q_1x^{m-1} + \cdots + q_m}, \tag{6.1}$$

where m and n are given, and $s(x)$ is a function continuous on $[a, b]$, so that the deviation of $Q(x)$ from a chosen continuous function $f(x)$,

$$H_Q = \max_{a \le x \le b} |f(x) - Q(x)| \tag{6.2}$$

shall be a minimum.

Chebyshev established the beautiful existence theorem [6; 2]:

> The function $P(x)$, which deviates least from the function $f(x)$ than does any other function of the type exemplified by equation (6.1) is completely characterized by the following property: If the function can be expressed in the form
>
> $$P(x) = s(x)\frac{a_0x^{n-\sigma} + a_1x^{n-\sigma-1} + \cdots + a_{n-\sigma}}{b_0x^{m-\tau} + b_1x^{m-\tau-1} + \cdots + b_{m-\tau}} = s(x)\frac{A(x)}{B(x)}$$
>
> where $0 \le \sigma \le n, 0 \le \tau \le m, b_0 \ne 0$ and the fraction $\frac{A(x)}{B(x)}$ is irreducible, then the number N of consecutive points of the interval $[a, b]$ at which the difference $f(x) - P(x)$, with alternate change of sign, takes on the value H_p, is not less than $m + n + 2 - d$, where $d = \min(\sigma, \tau)$; in case $P(x) \equiv 0$, then $N \ge n + 2$.

Chebyshev did not provide a constructive approach to the problem of finding the rational approximations whose existence is guaranteed by the above theorem. He, and E. Solotarev did work out one example, based on the theory of Jacobian elliptic functions, that meets the requirements of the theorem [16]. Since that time, though, many people have sought to obtain an explicit method of attack for determining these rational approximations [8; 9; 10]. The problem is especially complicated by the fact that the class of continuous functions is a very broad one. Most of the methods of attack that have been developed deal with a more restrictive class of functions: bounded variation, analytic, or the like.

A substantial advance was made by H. Padé in his now classic thesis of 1892 [13]. Padé's method, mentioned briefly at the end of the last section, yields excellent rational approximations of analytic functions by means of solutions of a system of linear algebraic equations [18]. The method is an extension of some earlier work of Frobenius [6]. However, it does not provide rational Chebyshev approximations. It is known that rational forms in Chebyshev polynomials yield better accuracy than ordinary rational forms [16].

Maehly gave a method for obtaining rational Chebyshev approximations of functions of bounded variation on the unit interval [12; 16]. It has the substantial disadvantage of requiring that the given function be first expanded in a series of Chebyshev polynomials. If the function is anywhere complicated, these expansions may be devilishly hard to obtain.

To the best of our knowledge, no method is known for obtaining rational Chebyshev approximations that is better, more direct, or more powerful than the one we have presented in this paper. The method was discovered by one of the authors (Castellanos) as a result of his work on formulas to approximate π while in preparation of "The Ubiquitous π," *Math. Magazine* **61.2-3** (April-June 1988). The delicate and time-consuming task of carrying the algorithm into a working computer program was done by the other author (Rosenthal).

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