THE RABBIT PROBLEM REVISITED

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INTRODUCTION

In *Liber Abaci* (1202), Leonardo da Pisa posed and solved the following problem.

> A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

The sequence obtained to solve this problem—the celebrated Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ...—appears in a large number of natural phenomena (see [2], [6]) and has natural applications in computer science (see [1]).

Here we reformulate the rabbit problem to recover two generalizations of the Fibonacci sequence presented elsewhere (see [7], [8]). Then, using a fixed-point technique, we present an elementary proof of the convergence of the sequences of ratios of two successive generalized Fibonacci numbers. The limits of these sequences will be called here *generalized golden numbers.* Finally, we reconsider electrical schemes to generate these ratios (see also [3]).

1. THE RABBIT PROBLEM REVISITED

The modifications to the rabbit problem we would like to consider here are the possibility that the mature rabbits produce more than one new pair of rabbits, and also the possibility of an increase in the productivity during the first few months. These two considerations lead to the following reformulation of the rabbit problem.

> A certain man put a pair of newborn male-female rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that each month
>
> (a) a i-month old pair of male-female rabbits gives birth to $(i-1)s$ pair(s) of male-female rabbits until it is r-months old, and
>
> (b) a more than r-month old pair of male-female rabbits continues to give birth to $(r-1)s$ pairs of male-female rabbits?

In this formulation it is assumed that s is a positive integer.

Let u_n be the total number of pairs of male-female rabbits at the n^{th} month, and v_n^i be the number of i-month old pairs of male-female rabbits at the n^{th} month. Since v_n^0 is the number of newborn pairs of male-female rabbits at the n^{th} month, we have

$$u_n = u_{n-1} + v_n^0 \quad \text{and} \quad v_n^i = v_{n-i}^0. \tag{1}$$

Then

$$v_n^0 = 0 \quad \text{for } n = -1, -2, -3, \ldots \tag{2a}$$

$$v_0^0 = 1 \tag{2b}$$

$$v_n^0 = \sum_{i=1}^{r}(i-1)sv_n^i + \sum_{i=r+1}^{+\infty}(r-1)sv_n^i \quad \text{for } n = 1, 2, 3, \ldots . \tag{2c}$$

Using (1), (2c) becomes

$$v_n^0 = s\sum_{i=2}^{r} u_{n-i},$$

and it follows that

$$u_n = 0 \quad \text{for } n = -1, -2, -3, \ldots \tag{3a}$$

$$u_0 = 1 \tag{3b}$$

$$u_n = u_{n-1} + s\sum_{i=2}^{r} u_{n-i} \quad \text{for } n = 1, 2, 3, \ldots . \tag{3c}$$

Remark 1: For $r = 2$ we have the *multi-nacci sequence of order s* recently considered by Levine [7]. One interesting property of these sequences is

$$u_n^2 - u_{n+1}u_{n-1} = (-s)^n.$$

Remark 2: For $s = 1$ we have the *r-generalized Fibonacci sequence* introduced by Miles [8] and also studied by Flores [4] and Dubeau [3].

From these two remarks, we can call a *r-generalized multi-nacci sequence of order s* the sequence of u_n's generated by (3).

2. CONVERGENCE OF RATIOS

In this section, we extend the method presented in [3] and [5] to obtain the limit of the sequence of ratios $t_n = u_n / u_{n-1}$ ($n = 1, 2, 3, \ldots$). Since the u_n's form an increasing sequence, we have $t_n \geq 1$ for $n = 1, 2, 3, \ldots$. From (3c) we have

$$t_n = 1 + s\sum_{i=2}^{r} \frac{u_{n-i}}{u_{n-1}} \quad (n \geq 1),$$

and using the definition of t_n, we obtain

$$t_n = \begin{cases} 1 + s\displaystyle\sum_{i=2}^{n} \frac{1}{\prod_{j=1}^{i-1} t_{n-j}} & n = 1, \ldots, r-1, \\ 1 + s\displaystyle\sum_{i=2}^{r} \frac{1}{\prod_{j=1}^{i-1} t_{n-j}} & n = r, r+1, r+2, \ldots . \end{cases}$$

The results of this section are then mainly based on the following two remarks.

Remark 3: t_n depends only on the preceding $r - 1$ values $t_{n-1}, t_{n-2}, \ldots, t_{n-(r-1)}$, and we can write $t_n = f(t_{n-1}, \ldots, t_{n-(r-1)})$.

Remark 4: If $t_{n-1}, \ldots, t_{n-(r-1)}$ are all greater than or equal to $b > 0$, then $t_n \leq f(b, \ldots, b)$ and if $t_{n-1}, \ldots, t_{n-(r-1)}$ are all less than or equal to $b > 0$, then $t_n \geq f(b, \ldots, b)$.

Let us use the function $f(\cdot,\dots,\cdot)$ to define another function $F(\cdot)$ as follows: $F(x) = f(x,\dots,x)$ or, explicitly,

$$F(x) = 1 + s\sum_{i=2}^{r}\frac{1}{x^{i-1}} \quad \text{for } x \neq 0. \tag{4}$$

The convergence result we look for will be obtained from the study of the function $F(\cdot)$. The next lemma summarizes the main properties of $F(\cdot)$.

Lemma 1: Let $s>0$, $r \in \{2,3,4,\dots\}$ and $x \neq 0$. Then

(a) $F(x) = \begin{cases} 1+s(r-1) & \text{if } x=1, \\ 1+\dfrac{s}{x^{r-1}}\,\dfrac{(x^{r-1}-1)}{(x-1)} & \text{if } x \neq 1; \end{cases}$

(b) $F(\cdot)$ is a strictly decreasing continuous convex function for $x>0$;

(c) $\lim_{x\to 0^+} F(x) = +\infty$ and $\lim_{x\to+\infty} F(x) = 1$;

(d) the equation $x = F(x)$ has a unique solution τ in the interval $(0,+\infty)$ and τ is the unique positive root of the polynomial

$$p(x) = x^r - x^{r-1} - s\sum_{i=2}^{r} x^{r-i}. \quad \square$$

Remarks 3 and 4 and the fact that $t_k \geq 1$ $(k \geq 1)$ suggest the construction of a sequence $\{b_\ell\}_{\ell=1}^{+\infty}$ such that

$$\begin{array}{lllll} b_1 = 1 & \leq & t_k & & \text{for } k \geq 1, \\ & & t_k & \leq \;F(b_1) = b_2 & \text{for } k \geq 1+(r-1), \\ b_3 = F(b_2) & \leq & t_k & & \text{for } k \geq 1+2(r-1), \\ & & t_k & \leq \;F(b_3) = b_4 & \text{for } k \geq 1+3(r-1), \\ b_5 = F(b_4) & \leq & t_k & & \text{for } k \geq 1+4(r-1), \\ & & \text{etc.} & & \end{array}$$

We have the following results about the sequence $\{b_\ell\}_{\ell=1}^{+\infty}$.

Lemma 2: Let $\{b_\ell\}_{\ell=1}^{+\infty}$ such that $b_1 = 1$ and $b_{\ell+1} = F(b_\ell)$ for $\ell = 1,2,3,\dots$, then

(a) the subsequence $\{b_{2\ell-1}\}_{\ell=1}^{+\infty}$ is strictly increasing and the subsequence $\{b_{2\ell}\}_{\ell=1}^{+\infty}$ is strictly decreasing;

(b) for all i and $j \geq 1$, we have $b_{2i-1} < b_{2j}$;

(c) there exists a positive constant $\beta < 1$ such that $0 < b_{2\ell+2} - b_{2\ell+1} < \beta^{2\ell} s(r-1)$ for $\ell = 1, 2, 3, \dots$;

(d) the sequence $\{b_\ell\}_{\ell=1}^{+\infty}$ converges to the unique positive real root of the polynomial

$$p(x) = x^r - x^{r-1} - s\sum_{i=2}^{r} x^{r-i}.$$

Proof: (a) and (b) follow from $1 = b_1 < b_2 = F(b_1)$, and if $0 < \alpha < \beta$, then $1 < F(\beta) < F(\alpha)$. To prove (c) we use (4) and consider

$$0 < b_{2\ell+2} - b_{2\ell+1} = F(b_{2\ell+1}) - F(b_{2\ell}) = s\sum_{i=2}^{r}\left(\frac{1}{b_{2\ell+1}^{i-1}} - \frac{1}{b_{2\ell}^{i-1}}\right)$$

$$= s\frac{(b_{2\ell} - b_{2\ell+1})}{b_{2\ell}b_{2\ell+1}}\sum_{i=0}^{r-2}\sum_{j=0}^{i}\frac{1}{b_{2\ell}^{j}b_{2\ell+1}^{i-j}} \le s\frac{(b_{2\ell} - b_{2\ell+1})}{b_{2\ell}b_{2\ell+1}}\left(\sum_{i=0}^{r-2}\frac{1}{b_{2\ell}^{i}}\right)\left(\sum_{i=0}^{r-2}\frac{1}{b_{2\ell+1}^{i}}\right).$$

But

$$\sum_{i=0}^{r-2}\frac{1}{b_{2\ell}^{i}} = \frac{(b_{2\ell+1} - 1)}{s}b_{2\ell} \quad\text{and}\quad \sum_{i=0}^{r-2}\frac{1}{b_{2\ell+1}^{i}} = \frac{(b_{2\ell+1}^{r-1} - 1)}{b_{2\ell+1}^{r-2}(b_{2\ell+1} - 1)},$$

then

$$0 < b_{2\ell+2} - b_{2\ell+1} \le (b_{2\ell} - b_{2\ell+1})\left(1 - \frac{1}{b_{2\ell+1}^{r-1}}\right).$$

Also, $1 \le b_k \le 1 + s(r-1)$, then $1 \le b_k^{r-1} \le [1 + s(r-1)]^{r-1}$, and it follows that

$$0 \le 1 - \frac{1}{b_k^{r-1}} < 1 - \frac{1}{[1 + s(r-1)]^{r-1}} = \beta < 1.$$

Hence, $0 \le b_{2\ell+2} - b_{2\ell+1} \le (b_{2\ell} - b_{2\ell+1})\beta$. Similarly, we can prove $0 < b_{2\ell} - b_{2\ell+1} \le (b_{2\ell} - b_{2\ell-1})\beta$, and we can conclude that

$$0 < b_{2\ell+2} - b_{2\ell+1} \le \beta^2(b_{2\ell} - b_{2\ell-1}) \le \cdots \le \beta^{2\ell}(b_2 - b_1).$$

But $b_2 - b_1 = s(r-1)$, and the result follows. Finally, from (c), the upperbounded increasing subsequence $\{b_{2\ell+1}\}_{\ell=1}^{+\infty}$ and the lower bounded decreasing subsequence $\{b_{2\ell}\}_{\ell=1}^{+\infty}$ both converge to the value τ defined in Lemma 1. □

Figure 1,on the following page, describes the construction and the convergence of the sequence $\{b_\ell\}_{\ell=0}^{+\infty}$.

We are now ready to prove the main results.

Theorem 1: Let $s > 0, r \in \{2, 3, 4, 5, \ldots\}, u_n$ as given by (3), and $t_n = u_n / u_{n-1}$ for $n \ge 1$. The sequence $\{t_n\}_{n=1}^{+\infty}$ converges to the unique positive root τ of the polynomial

$$p(x) = x^r - x^{r-1} - s\sum_{i=2}^{r}x^{r-i}.$$

Proof: From the way the sequences $\{t_k\}_{k=1}^{+\infty}$ and $\{b_\ell\}_{\ell=1}^{+\infty}$ are generated, we have

$$t_k \ge b_{2\ell+1} \text{ for } k \ge 1 + 2\ell(r-1) \quad\text{and}\quad t_k \le b_{2\ell} \text{ for } k \ge 1 + (2\ell+1)(r-1).$$

The result follows from Lemma 2. □

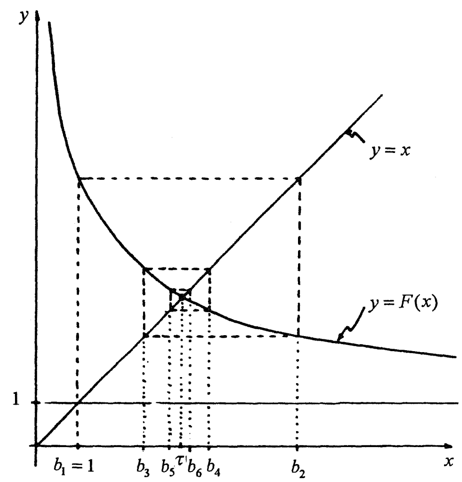

FIGURE 1. Graph of $y = F(x)$

Theorem 2: Let τ be considered as a function of τ and s. Then

(a) for any fixed $s > 0$, we have

(i) $\tau = \dfrac{1+\sqrt{1+4s}}{2}$ for $r = 2$,

(ii) τ increases as r increases,

and

(iii) $\lim\limits_{r\to+\infty} = 1+\sqrt{s}$;

(b) for any fixed τ, $\tau \sim \sqrt{s}$ for large s.

Proof: For $r = 2$, τ is the unique positive root of $p(x) = x^2 - x - s$, which corresponds to the given formula. Because $F(x)$ increases as r increases for fixed x and s, τ increases as r increases. Also

$$F(1+\sqrt{s}) = 1+\sqrt{s} - \frac{\sqrt{s}}{(1+\sqrt{s})^{r-1}} < 1+\sqrt{s},$$

then

$$1+\sqrt{s} - \frac{\sqrt{s}}{(1+\sqrt{s})^{r-1}} < \tau < 1+\sqrt{s}$$

and $\lim_{r\to+\infty} \tau = 1+\sqrt{s}$. Also, from those formulas and inequalities, we obtain $\tau \sim \sqrt{s}$ when s is large. □

The table below presents values of τ for some r and s. The last line of this table for $r = +\infty$ indicates $\lim_{r\to+\infty} \tau = 1+\sqrt{s}$.

When $r = 2$ and $s = 1$, τ corresponds to the *golden number.* For $s = 1$ and $r \in \{2, 3, \ldots\}$, τ has been called the *r-generalized golden number.* Hence, for $s > 0$ and $r \in \{2, 3, \ldots\}$, we could call τ the *r-generalized golden number of order s.*

Table of τ Values for Given r and s

r \ s	1	2	3	4	5
2	1.6180340	2.0000000	2.3027756	2.5615528	2.7912878
3	1.8392868	2.2695308	2.5986745	2.8751298	3.1179423
4	1.9275620	2.3593041	2.6868102	2.9611061	3.2017404
5	1.9659482	2.3924637	2.7160633	2.9874051	3.2257176
6	1.9835828	2.4054051	2.7262912	2.9958519	3.2328999
7	1.9919642	2.4106054	2.7299574	2.9986240	3.2350925
8	1.9960312	2.4127271	2.7312869	2.9995422	3.2357669
9	1.9980295	2.4135994	2.7317715	2.9998475	3.2359750
10	1.9990186	2.4139595	2.7319486	2.9999492	3.2360392
11	1.9995104	2.4141084	2.7320134	2.9999831	3.2360591
12	1.9997555	2.4141700	2.7320371	2.9999944	3.2360652
13	1.9998778	2.4141955	2.7320458	2.9999981	3.2360671
14	1.9999389	2.4142061	2.7320490	2.9999994	3.2360677
15	1.9999695	2.4142105	2.7320501	2.9999998	3.2360679
16	1.9999847	2.4142123	2.7320506	2.9999999	3.2360680
17	1.9999924	2.4142130	2.7320507	3.0000000	3.2360680
18	1.9999962	2.4142133	2.7320508	3.0000000	3.2360680
19	1.9999981	2.4142135	2.7320508	3.0000000	3.2360680
20	1.9999990	2.4142135	2.7320508	3.0000000	3.2360680
21	1.9999995	2.4142135	2.7320508	3.0000000	3.2360680
22	1.9999998	2.4142136	2.7320508	3.0000000	3.2360680
23	1.9999999	2.4142136	2.7320508	3.0000000	3.2360680
24	1.9999999	2.4142136	2.7320508	3.0000000	3.2360680
25	2.0000000	2.4142136	2.7320508	3.0000000	3.2360680
$+\infty$	2.	2.4142136	2.7320508	3.	3.2360680

3. ELECTRICAL SCHEMES

The method presented in [3] to generate the sequences of ratios $\{u_n / u_{n-1}\}_{n=1}^{+\infty}$ using electrical schemes can also be used here. Indeed, if

$$\Omega_{j,i} = \frac{u_{j+i}}{u_j} = \frac{u_{j+i-1} + s\sum_{k=2}^{r} u_{j+i-k}}{u_j} = \Omega_{j,i-1} + s\sum_{k=2}^{r} \Omega_{j,i-k}, \tag{5}$$

which correspond to $1 + s(r-1)$ resistances connected in series. Also

$$\Omega_{j,i} = \frac{u_{j+i}}{u_j} = \frac{u_{j+i}}{u_{j-1} + s\sum_{k=2}^{r} u_{j-k}} = \frac{1}{\frac{1}{u_{j+i}/u_{j-1}} + s\sum_{k=2}^{r}\frac{1}{u_{j+i}/u_{j-k}}} = \frac{1}{\frac{1}{\Omega_{j-1,i+1}} + s\sum_{k=2}^{r}\frac{1}{\Omega_{j-k,i+k}}}, \tag{6}$$

which correspond to $1 + s(r-1)$ resistances connected in parallel. Here, again, it is assumed that s is a positive integer.

Those two formulas, (5) and (6), suggest the following process to generate the resistances $\Omega_{j,i}$ ($j = 0, 1, 2, \ldots,$ and $i = -(r-1, \ldots, -1, 0, 1, \ldots, r-1)$:

(a) generate $\Omega_{j,i}$ ($i = -(r-1), \ldots, -1$) using (6) with $\Omega_{j-1,i+1}$ and s of each $\Omega_{j-k,i+k}$ for $k =$ 2, ..., r;

(b) $\Omega_{j,0} = 1$;

(c) gernerate $\Omega_{j,i}$ ($i = 1, 2, 3, \ldots, r-1$) using (5) with $\Omega_{j,i-1}$ and s of each $\Omega_{j,i-k}$ for $k = 2$, ..., r.

Note that the ratios we are interested in correspond to $\Omega_{j,1}$ ($j = 0, 1, 2, 3, \ldots$).

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AMS numbers: 11B39, 05A10, 94C05

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ERRATUM FOR "COMPLEX FIBONACCI AND LUCAS NUMBERS, CONTINUED FRACTIONS, AND THE SQUARE ROOT OF THE GOLDEN RATIO"

The Fibonacci Quarterly **31.1** (1993):7-20

It has been pointed out to me by a correspondent who wished to remain anonymous that the number 185878941, which was printed in the "loose ends" Section 7 on page 19 of the paper, has a factor 3. This, however, was a misprint for 285878941, which is $(\ell_{19}^2 + \ell_{19}'^2)/2$, and the same correspondent has checked that this is a prime by using Mathematica. The misprint was important because it appeared to undermine one of the interesting conjectures on that page (and incidentally calls into question my ability to "cast out 3s"!). The same correspondent pointed out that $34227121 = 137 \times 249833$.

I. J. Good