SECOND DERIVATIVE SEQUENCES OF FIBONACCI AND LUCAS POLYNOMIALS

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1. INTRODUCTION AND GENERALITIES

Let us consider the *Fibonacci polynomials* $U_n(x)$ and the *Lucas polynomials* $V_n(x)$ (or simply U_n and V_n, when no misunderstanding can arise) defined by the second-order linear recurrence relations

$$U_n = xU_{n-1} + U_{n-2} (U_0 = 0, U_1 = 1), \tag{1.1}$$

and

$$V_n = xV_{n-1} + V_{n-2} \ (V_0 = 2, V_1 = x), \tag{1.2}$$

where x is an indeterminate. It is well known that the polynomials U_n and V_n, can be expressed by means of the *Binet forms*

$$U_n = (\alpha^n - \beta^n)/\Delta \tag{1.3}$$

and

$$V_n = \alpha^n + \beta^n, \tag{1.4}$$

where

$$\begin{aligned} \Delta &= \sqrt{x^2+4} \\ \alpha &= (x+\Delta)/2 \\ \beta &= (x-\Delta)/2 = -1/\alpha = x - \alpha. \end{aligned} \tag{1.5}$$

Recall that further expressions for U_n and V_n, (e.g., see [1], [3]) are

$$U_n = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-j}{j} x^{n-1-2j} \ (n \geq 1) \tag{1.6}$$

and

$$V_n = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n}{n-j} \binom{n-j}{j} x^{n-2j} \quad (n \geq 1) \tag{1.7}$$

where $\lfloor a \rfloor$ denotes the greatest integer not exceeding a.

In [4] we considered the numbers $F_n^{(1)}$ and $L_n^{(1)}$ obtainable by taking the first derivative of the polynomials (1.6) and (1.7) at $x = 1$, and studied their properties. The basic results established in [4] are

$$F_n^{(1)} = \left[\frac{d}{dx} U_n(x) \right]_{x=1} = (nL_n - F_n)/5 \tag{1.8}$$

$$L_n^{(1)} = \left[\frac{d}{dx}V_n(x)\right]_{x=1} = nF_n, \tag{1.9}$$

where F_n and L_n are the usual Fibonacci and Lucas numbers, respectively. Observe that the numbers $F_n^{(1)}$ and $L_n^{(1)}$ are, respectively, denoted by F_n' and L_n' in [4].

In this paper we consider the second derivative with respect to x of the polynomials (1.6) and (1.7) and investigate some of their properties, thus keeping, in part, the promise made to the reader in section 4 of [4]. In the concluding section, we offer a brief glimpse of the implications of investigating the k^{th} derivatives of $U_n(x)$ and $V_n(x)$

1.1 Definitions

Let us define the polynomials $U_n^{(2)}$ and $V_n^{(2)}$, which are also obtainable from (1.6) and (1.7), as

$$U_n^{(2)} = \frac{d^2}{dx^2}U_n = \sum_{j=0}^{\lfloor (n-3)/2 \rfloor}(n-1-2j)(n-2-2j)\binom{n-1-j}{j}x^{n-3-2j} \quad (n \geq 1), \tag{1.10}$$

and

$$V_n^{(2)} = \frac{d^2}{dx^2}V_n = \sum_{j=0}^{\lfloor (n-2)/2 \rfloor}\frac{n(n-2j)(n-1-2j)}{n-j}\binom{n-j}{j}x^{n-2-2j} \quad (n \geq 1). \tag{1.11}$$

Observe that

$$U_0^{(2)} = V_0^{(2)} = 0 \text{ [from (1.1) and (1.2)]} \tag{1.12}$$

and

$$U_1^{(2)} = U_2^{(2)} = V_1^{(2)} = 0 \tag{1.12'}$$

according to the convention that a sum vanishes whenever the upper range indicator is less than the lower one. From (1.10)-(1.12′) we can write the first few elements of the sequences $\{U_n^{(2)}\}_0^\infty$ and $\{V_n^{(2)}\}_0^\infty$, namely,

$$\begin{array}{l|l}
U_0^{(2)} = U_1^{(2)} = U_2^{(2)} = 0 & V_0^{(2)} = V_1^{(2)} = 0 \\
U_3^{(2)} = 2 & V_2^{(2)} = 2 \\
U_4^{(2)} = 6x & V_3^{(2)} = 6x \\
U_5^{(2)} = 12x^2 + 6 & V_4^{(2)} = 12x^2 + 8 \\
U_6^{(2)} = 20x^3 + 24x & V_5^{(2)} = 20x^3 + 30x \\
U_7^{(2)} = 30x^4 + 60x^2 + 12 & V_6^{(2)} = 30x^4 + 72x^2 + 18 \\
U_8^{(2)} = 42x^5 + 120x^3 + 60x & V_7^{(2)} = 42x^5 + 140x^3 + 84x \\
U_9^{(2)} = 56x^6 + 210x^4 + 180x^2 + 20 & V_8^{(2)} = 56x^6 + 240x^4 + 240x^2 + 32 \\
U_{10}^{(2)} = 72x^7 + 336x^5 + 420x^3 + 120x & V_9^{(2)} = 72x^7 + 378x^5 + 540x^3 + 180x \\
 & V_{10}^{(2)} = 90x^8 + 560x^6 + 1050x^4 + 600x^2 + 50.
\end{array} \tag{1.13}$$

In this paper we confine ourselves to studying some properties of the above sequences for the case $x = 1$. Since, letting $x = 1$ in (1.1)-(1.5), we have the usual Fibonacci and Lucas numbers, the sequences of integers $\{U_n^{(2)}(1)\}$ and $\{V_n^{(2)}(1)\}$ will be denoted by $\{F_n^{(2)}\}$ and $\{L_n^{(2)}\}$ and defined as *Fibonacci* and *Lucas second derivative sequences*, respectively.

From (1.13), the first few values of $F_n^{(2)}$ and $L_n^{(2)}$ are

n	0	1	2	3	4	5	6	7	8	9	10
$F_n^{(2)}$	0	0	0	2	6	18	44	102	222	466	948
$L_n^{(2)}$	0	0	2	6	20	50	120	266	568	1170	2350

(1.14)

A large number of relationships involving $F_n^{(2)}$, $L_n^{(2)}$, $F_n^{(1)}$, $L_n^{(1)}$, F_n and L_n will be exhibited in the following sections. Their proofs are not very complicated but they are rather lengthy, so, for the sake of brevity, only some of them will be given in full detail.

2. EXPRESSIONS FOR $F_n^{(2)}$ AND $L_n^{(2)}$ IN TERMS OF FIBONACCI AND LUCAS NUMBERS

Expressions for $F_n^{(2)}$ and $L_n^{(2)}$ in terms of U_n and V_n can be obtained from the definitions (1.10) and (1.11) and the Binet forms (1.3)-(1.5). Letting the bracketed superscript $(^k)$ denote the k^{th} derivative with respect to x and taking into account the results established in section 2 of [4], we can write

$$\begin{aligned} U_n^{(2)} &= \frac{d^2}{dx^2}\frac{\alpha^n-\beta^n}{\Delta} = \frac{d}{dx}U_n^{(1)} = \frac{d}{dx}\frac{n(\alpha^n+\beta^n)\Delta - x(\alpha^n-\beta^n)}{\Delta^3} \\ &= \frac{[n(\alpha^n+\beta^n)\Delta - x(\alpha^n-\beta^n)]^{(1)}\Delta^3 - (\Delta^3)^{(1)}[n(\alpha^n+\beta^n)\Delta - x(\alpha^n-\beta^n)]}{\Delta^6} \\ &= \frac{[(n^2-1)\Delta U_n]\Delta^3 - 3x\Delta[n\Delta V_n - x\Delta U_n]}{\Delta^6} = \frac{[(n^2-1)\Delta^2+3x^2]U_n - 3nxV_n}{\Delta^4}. \end{aligned} \tag{2.1}$$

Analogously, we have

$$\begin{aligned} V_n^{(2)} &= \frac{d^2}{dx^2}(\alpha^n+\beta^n) = \frac{d}{dx}V_n^{(1)} = \frac{d}{dx}\frac{n(\alpha^n-\beta^n)}{\Delta} \\ &= n\frac{[(\alpha^n)^{(1)}-(\beta^n)^{(1)}]\Delta - \Delta^{(1)}(\alpha^n-\beta^n)}{\Delta^2} \\ &= n\frac{n\alpha^n+n\beta^n - x(\alpha^n-\beta^n)/\Delta}{\Delta^2} = \frac{n(nV_n - xU_n)}{\Delta^2}. \end{aligned} \tag{2.2}$$

Letting $x = 1$ in (2.1) and (2.2) yields

$$F_n^{(2)} = \frac{(5n^2-2)F_n - 3nL_n}{25} \tag{2.3}$$

and

$$L_n^{(2)} = \frac{n(nL_n - F_n)}{5}, \tag{2.4}$$

whence the expressions for negative-subscripted elements of the Fibonacci and Lucas second derivative sequences can be easily deduced, namely,

$$F_{-n}^{(2)} = (-1)^{n+1} F_n^{(2)} \tag{2.5}$$

and

$$L_{-n}^{(2)} = (-1)^n L_n^{(2)}. \tag{2.6}$$

Observe that, from (1.8), (1.9), (2.3), and (2.4), we get the following equivalent expressions for $F_n^{(2)}$ and $L_n^{(2)}$:

$$F_n^{(2)} = (nL_n^{(1)} - 3F_n^{(1)} - F_n) / 5, \tag{2.7}$$

and

$$L_n^{(2)} = nF_n^{(1)}. \tag{2.8}$$

3. SOME IDENTITIES INVOLVING THE NUMBERS $F_n^{(2)}$ AND $L_n^{(2)}$

Some simple properties of the numbers $F_n^{(2)}$ and $L_n^{(2)}$ can be derived from (1.8), (1.9), and (2.3)-(2.8). First, let us state the following four identities.

Identity 1: $F_{n+m}^{(2)} + (-1)^m F_{n-m}^{(2)} = L_m F_n^{(2)} + F_n L_m^{(2)} + 2mF_m F_n^{(1)}$.

Identity 2: $F_{n+m}^{(2)} - (-1)^m F_{n-m}^{(2)} = F_m L_n^{(2)} + L_n F_m^{(2)} + 2nF_n F_m^{(1)}$.

Identity 3: $L_{n+m}^{(2)} + (-1)^m L_{n-m}^{(2)} = L_m L_n^{(2)} + L_n L_m^{(2)} + 2L_n^{(1)} L_m^{(1)}$.

Identity 4: $L_{n+m}^{(2)} - (-1)^m L_{n-m}^{(2)} = nL_n F_m^{(1)} + mL_m F_n^{(1)} + (n^2 + m^2) F_n F_m$.

For the sake of brevity, we shall prove only Identity 1.

Proof of Identity 1: From (2.3) we write

$$\begin{aligned} F_{n+m}^{(2)} + (-1)^m F_{n-m}^{(2)} &= \{[5(n+m)^2 - 2]F_{n+m} - 3(n+m)L_{n+m} \\ &\quad + (-1)^m[5(n-m)^2 - 2]F_{n-m} - 3(-1)^m (n-m) L_{n-m}\} / 25 \\ &= \{[5(n^2+m^2) - 2][F_{n+m} + (-1)^m F_{n-m}] + 10nm[F_{n+m} - (-1)^m F_{n-m}] \\ &\quad - 3n[L_{n+m} + (-1)^m L_{n-m}] - 3m[L_{n+m} - (-1)^m L_{n-m}]\} / 25. \end{aligned} \tag{3.1}$$

After some manipulations involving the use of (2.3), (2.4), (1.8), and the identities I_{21} - I_{24} [5, page 59] a compact form of which is

$$\begin{aligned} F_{h+k} + (-1)^k F_{h-k} &= F_h L_k \\ F_{h+k} - (-1)^k F_{h-k} &= L_h F_k, \end{aligned}$$

the identity (3.1) can be rewritten as

$$\begin{aligned} F_{n+m}^{(2)} + (-1)^m F_{n-m}^{(2)} &= [5(n^2+m^2) F_n L_m + 10nmF_m L_n - 2F_n L_m - 3nL_n L_m - 15mF_n F_m] / 25 \\ &= L_m[(5n^2 - 2)F_n - 3nL_n] / 25 + mF_n(mL_m - 3F_m) / 5 + 2nmF_m L_n / 5 \\ &= L_m F_n^{(2)} + mF_n(mL_m - F_m) / 5 - 2mF_n F_m / 5 + 2nmF_m L_n / 5 \\ &= L_m F_n^{(2)} + F_n L_m^{(2)} + 2mF_m(nL_n - F_n) / 5 \\ &= L_m F_n^{(2)} + F_n L_m^{(2)} + 2mF_m F_n^{(1)}. \quad \square \end{aligned}$$

Particular cases of Identities 1-4 are

Identity 5 ($m=1$ in Id. 2): $F_{n-1}^{(2)}+F_{n+1}^{(2)}=L_n^{(2)}$.

Identity 6 ($m=1$ in Id. 4): $L_{n-1}^{(2)}+L_{n+1}^{(2)}=F_n^{(1)}+(n^2+1)F_n=F_n^{(1)}+nL_n^{(1)}+F_n$.

Identity 7 ($m=2$ in Id. 2): $F_{n+2}^{(2)}-F_{n-2}^{(2)}=L_n^{(2)}+2L_n^{(1)}$.

Identity 8 ($n=m$ in Id. 2): $F_{2m}^{(2)}=3F_mL_m^{(2)}+L_mF_m^{(2)}$.

Identity 9 ($n=m$ in Id. 3): $L_{2m}^{(2)}=2[L_mL_m^{(2)}+(L_m^{(1)})^2]$.

Identity 10 ($n=2m$ in Id. 2): $F_{3m}^{(2)}=F_m[L_{2m}^{(2)}+4mL_mF_m^{(1)}]+[L_{2m}+(-1)^m]F_m^{(2)}$.

Identity 11 ($n=2m$ in Id. 3): $L_{3m}^{(2)}=3\{L_m^{(2)}[L_{2m}+(-1)^m]+2L_m(L_m^{(1)})^2\}$.

Next, we derive

Identity 12: $F_n^{(1)}L_n^{(2)}-L_n^{(1)}F_n^{(2)}=[F_n(5L_n^{(2)}+4L_n^{(1)})+4(-1)^nn^3]/25$.

Proof: From (1.8), (1.9), (2.7), and (2.8), we have

$$F_n^{(1)}L_n^{(2)}-L_n^{(1)}F_n^{(2)}=[5n(F_n^{(1)})^2-n(L_n^{(1)})^2+3F_n^{(1)}L_n^{(1)}+F_nL_n^{(1)}]/5. \tag{3.2}$$

Using the identities

$$(F_n^{(1)})^2=(n^2L_n^2+F_n^2-2nF_{2n})/25, \tag{3.3}$$

$$(L_n^{(1)})^2=n^2F_n^2, \tag{3.4}$$

$$F_n^{(1)}L_n^{(1)}=n(nF_{2n}-F_n^2)/5, \tag{3.5}$$

$$F_nL_n^{(1)}=nF_n^2, \tag{3.6}$$

and the identity I_{12} [5, page 56] [namely, $5F_k^2=L_k^2-4(-1)^k$], we find that (3.2) becomes

$$\begin{aligned}F_n^{(1)}L_n^{(2)}-L_n^{(1)}F_n^{(2)}&=\left(\frac{n^3L_n^2+nF_n^2-2n^2F_{2n}}{5}-n^3F_n^2+\frac{3n^3F_{2n}-3nF_n^2}{5}+nF_n^2\right)\Big/5\\&=[n^3(L_n^2-5F_n^2)+n^2F_{2n}+3nF_n^2]/25=[4(-1)^nn^3+n^2F_{2n}+3nF_n^2]/25\\&=[nF_n(nL_n+3F_n)+4(-1)^nn^3]/25=[5F_nL_n^{(2)}+4nF_n^2+4(-1)^nn^3]/25\\&=[F_n(5L_n^{(2)}+4L_n^{(1)})+4(-1)^nn^3]/25. \quad \square\end{aligned}$$

Let us conclude this section by giving the *Simson formula* analogs for $F_n^{(2)}$ and $L_n^{(2)}$.

Identity 13: $(F_n^{(2)})^2-F_{n-1}^{(2)}F_{n+1}^{(2)}=\dfrac{2n^2L_{2n}-6nF_{2n}+8F_n^2-n^2(-1)^n(5n^2-13)}{125}$.

Identity 14: $(L_n^{(2)})^2-L_{n-1}^{(2)}L_{n+1}^{(2)}=\dfrac{2n^2L_{2n}-2nF_{2n}-4F_n^2+5n^2(-1)^n(n^2-1)}{25}$.

Proof of Identity 14: Using (2.4) and identities I_{19}, I_{20} [5, page 59],

$$F_{h-k}F_{h+k} - F_n^2 = (-1)^{h+k+1}F_k^2$$

$$L_{h-k}L_{h+k} - L_h^2 = 5(-1)^{h+k}F_k^2,$$

we can write

$$\begin{aligned}(L_n^{(2)})^2 - L_{n-1}^{(2)}L_{n+1}^{(2)} &= n^2(nL_n - F_n)^2/25 - (n^2-1)[(n-1)L_{n-1} - F_{n-1}][(n+1)L_{n+1} - F_{n+1}]/25 \\ &= n^2(n^2L_n^2 + F_n^2 - 2nF_{2n})/25 - (n^2-1)\{(n^2-1)[L_n^2 - 5(-1)^n] \\ &\quad - (n-1)[F_{2n} - (-1)^n] - (n+1)[F_{2n} + (-1)^n] + F_n^2 + (-1)^n\}/25. \end{aligned} \tag{3.7}$$

After some manipulations involving the use of I_{12} [5, page 56] and the identities I_{15}, I_{18} [5, page 59] a compact form of which is $L_{2h} + 2(-1)^h = L_h^2$, the identity (3.7) can be rewritten as

$$\begin{aligned}(L_n^{(2)})^2 - L_{n-1}^{(2)}L_{n+1}^{(2)} &= [(2n^2-1)L_n^2 - 2nF_{2n} + F_n^2 + (-1)^n(5n^4 - 9n^2 + 4)]/25 \\ &= [2n^2L_{2n} - 2nF_{2n} + F_n^2 - L_n^2 + 4(-1)^n + 5n^2(-1)^n(n^2-1)]/25 \\ &= [2n^2L_{2n} - 2nF_{2n} - 4F_n^2 + 5n^2(-1)^n(n^2-1)]/25. \quad \square \end{aligned}$$

Simson formula analogs for $U_n^{(2)}$ and $V_n^{(2)}$ may be obtained from (2.1) and (2.2), but their discovery is left to the perseverance of the reader.

4. SOME SIMPLE CONGRUENCE PROPERTIES OF $F_n^{(2)}$ AND $L_n^{(2)}$

Letting $m = 1$ in Identity 1 and Identity 3, we obtain

$$F_{n+1}^{(2)} - F_{n-1}^{(2)} = F_n^{(2)} + 2F_n^{(1)} \tag{4.1}$$

and

$$L_{n+1}^{(2)} - L_{n-1}^{(2)} = L_n^{(2)} + 2L_n^{(1)}, \tag{4.2}$$

respectively. From (4.1) and (4.2), the *recurrence relations*

$$F_n^{(2)} = F_{n-1}^{(2)} + F_{n-2}^{(2)} + 2F_{n-1}^{(1)} \quad (F_0^{(2)} = F_1^{(2)} = 0) \tag{4.3}$$

and

$$L_n^{(2)} = L_{n-1}^{(2)} + L_{n-2}^{(2)} + 2L_{n-1}^{(1)} \quad (L_0^{(2)} = L_1^{(2)} = 0) \tag{4.4}$$

can be readily obtained, where the initial conditions have been taken from (1.14). The relations (4.3) and (4.4) allow us to state the following proposition.

Proposition 1: $F_n^{(2)}$ and $L_n^{(2)}$ are even for all n.

Further congruence properties of $F_n^{(2)}$ and $L_n^{(2)}$ can be easily established.

Proposition 2: $F_n^{(2)} \equiv 0 \pmod 6$ for $n \equiv 0, \pm 1, \pm 2, \pm 4, \pm 5 \pmod{12}$.

Proposition 3: $L_n^{(2)} \equiv 0 \pmod 6$ for $n \equiv 0 \pmod 3$ or $n \equiv \pm 1 \pmod{12}$.

Proposition 4: $L_n^{(2)} \equiv 0 \pmod{10}$ for $n \equiv 0, \pm 1 \pmod 5$.

The proofs of Propositions 2-4 are similar, so, for the sake of brevity, we shall prove only Proposition 3.

Proof of Proposition 3: From (2.4) and Proposition 1, it is apparent that we have to find conditions for $n(nL_n - F_n)$ to be divisible by 3. The first condition is trivial: $n \equiv 0 \pmod 3$. The second condition is given by the solution of the congruence $nL_n \equiv F_n \pmod 3$. The repetition period of the sequences $\{\langle F_n\rangle_3\}$ and $\{\langle L_n\rangle_3\}$ (the Fibonacci and Lucas sequences reduced modulo 3) is 8 (see [2, page 55]), whereas the repetition period of the sequence of naturals reduced modulo 3 is 3. Since l.c.m.(3, 8) = 24, we have to inspect the elements of the sequences $\{\langle nL_n\rangle_3\}_0^{23}$ and $\{\langle F_n\rangle_3\}_0^{23}$ and look for the equality

$$\langle nL_n\rangle_3 = \langle F_n\rangle_3. \tag{4.5}$$

It is readily seen that (4.5) is fulfilled for $n \equiv 0, \pm 1 \pmod{12}$. □

5. EVALUATION OF SOME SERIES INVOLVING $F_n^{(2)}$ AND $L_n^{(2)}$

In this section, several finite series involving $F_n^{(2)}$ and $L_n^{(2)}$ are considered and closed form expressions for their sums are exhibited. For the sake of brevity, only a few among them are proved in detail by using some results obtained in [4] and the further identities

$$\sum_{i=0}^{n} i(-1)^i F_{n-2i} = -(nL_{n+1} + 2F_n)/5 = -F_{n+1}^{(1)}, \tag{5.1}$$

$$\sum_{i=0}^{n} i(-1)^i L_{n-2i} = nF_{n+1} = L_{n+1}^{(1)} - F_{n+1}, \tag{5.2}$$

$$\sum_{i=0}^{n} F_i F_{n-i} = (nL_n - F_n)/5 = F_n^{(1)}, \tag{5.3}$$

$$\sum_{i=0}^{n} F_i L_{n-i} = (n+1)F_n = L_n^{(1)} + F_n. \tag{5.4}$$

The proofs of (5.1)-(5.4) can be carried out with the aid of the Binet forms (1.3)-(1.5) and [4, (3.1)]. Since they are rather tedious, they are omitted in this context.

5.1. Results

The following results have been obtained.

Proposition 5: $\sum_{i=0}^{n} F_i^{(2)} = F_{n+2}^{(2)} - 2(F_{n+3}^{(1)} - F_{n+4} + 1).$

Proposition 6: $\sum_{i=0}^{n} L_i^{(2)} = L_{n+2}^{(2)} - 2(L_{n+3}^{(1)} - L_{n+4} + 2).$

Proposition 7: $\sum_{i=0}^{n} \binom{n}{i} F_i^{(2)} = [5n^2 F_{2n-2} - (3n+2)F_{2n} + nF_{2n-7}]/25.$

Proposition 8: $\sum_{i=0}^{n}\binom{n}{i}L_i^{(2)} = n[(n-1)L_{2n-2} + 2F_{2n-2}]/5.$

We point out that several equivalent expressions for the above sums can be given. For example, we have

Proposition 8': $\sum_{i=0}^{n}\binom{n}{i}L_i^{(2)} = L_{n-1}(L_{n-1}^{(2)} + F_{n-1}^{(1)}) + n[3F_{2n-2} + 2(n-1)(-1)^n]/5.$

Finally, the following convolution identities have been established.

Proposition 9: $\sum_{i=0}^{n} F_i^{(1)}F_{n-i} = \frac{1}{2}F_n^{(2)}.$

Proposition 10: $\sum_{i=0}^{n} L_i^{(1)}F_{n-i} = \frac{1}{2}L_n^{(2)}.$

Proposition 11: $\sum_{i=0}^{n} F_i^{(1)}L_{n-i} = \frac{1}{2}L_n^{(2)} + F_n^{(1)}.$

Proposition 12: $\sum_{i=0}^{n} L_i^{(1)}L_{n-i} = \frac{5}{2}F_n^{(2)} + 2F_n^{(1)} + L_n^{(1)} + F_n.$

5.2 Proofs

Proof of Proposition 5: From (2.7), (1.8), and (1.9), we have

$$A_n = \sum_{i=0}^{n} F_i^{(2)} = \frac{1}{5}\left(\sum_{i=0}^{n} iL_i^{(1)} - 3\sum_{i=0}^{n} F_i^{(1)} - \sum_{i=0}^{n} F_i\right) = \frac{1}{5}\left(\sum_{i=0}^{n} i^2F_i - \frac{3}{5}\sum_{i=0}^{n} iL_i - \frac{2}{5}\sum_{i=0}^{n} F_i\right). \tag{5.5}$$

Using the Binet forms (1.3)-(1.5) (with $x = 1$), [4, (3.1) and (3.2)] and identity I_1 [5, page 52]

$$\sum_{i=1}^{k} F_i = F_{k+2} - 1,$$

we find that (5.5) becomes

$$A_n = \frac{1}{5}\left[\frac{1}{\sqrt{5}}\left(\sum_{i=0}^{n} i^2\alpha^i - \sum_{i=0}^{n} i^2\beta^i\right) - \frac{3}{5}\left(\sum_{i=0}^{n} i\alpha^i + \sum_{i=0}^{n} i\beta^i\right) - \frac{2}{5}(F_{n+2} - 1)\right]$$

$$= \frac{1}{5}\left[\frac{1}{\sqrt{5}}\left(\frac{n^2\alpha^{n+3} - (2n^2+2n-1)\alpha^{n+2} + (n+1)^2\alpha^{n+1} - \alpha^2 - \alpha}{-\beta^3}\right.\right.$$
$$\left. - \frac{n^2\beta^{n+3} - (2n^2+2n-1)\beta^{n+2} + (n+1)^2\beta^{n+1} - \beta^2 - \beta}{-\alpha^3}\right)$$
$$\left. - \frac{3}{5}\left(\frac{n\alpha^{n+2} - (n+1)\alpha^{n+1} + \alpha}{\beta^2} + \frac{n\beta^{n+2} - (n+1)\beta^{n+1} + \beta}{\alpha^2}\right)\right]$$

$$=\frac{1}{5}\left\{n^2F_{n+6}-(2n^2+2n-1)F_{n+5}+(n+1)^2F_{n+4}-8-\frac{3}{5}[nL_{n+4}-(n+1)L_{n+3}+4]-\frac{2}{5}(F_{n+2}-1)\right\}$$

$$=\frac{1}{5}\left[-n^2F_{n+3}-2nF_{n+3}+F_{n+6}+n^2F_{n+4}-\frac{3}{5}(nL_{n+2}-L_{n+3})-\frac{2}{5}F_{n+2}-10\right]$$

$$=\frac{1}{5}\left[n^2F_{n+2}-2nF_{n+3}+F_{n+6}-\frac{3}{5}(nL_{n+2}-2L_{n+2}-F_{n+2})-F_{n+2}+\frac{3}{5}(L_{n+3}+2L_{n+2})-10\right]$$

$$=\frac{1}{5}\left\{(n^2-1)F_{n+2}-2nF_{n+3}+F_{n+6}-\frac{3}{5}[(n+2)L_{n+2}-F_{n+2}]+3F_{n+3}-10\right\}$$

$$=\frac{1}{5}\left[(n^2-1)F_{n+2}-2nF_{n+3}+3F_{n+3}+F_{n+6}-3F_{n+2}^{(1)}-10\right]$$

$$=\frac{1}{25}\left[(5n^2-2)F_{n+2}-3nL_{n+2}-10nF_{n+3}-6L_{n+2}+5(3F_{n+3}+F_{n+6})-50\right]. \tag{5.6}$$

The equality (5.6) can be rewritten as

$$A_n=\frac{1}{25}\left\{\left[5(n+2)^2-2\right]F_{n+2}-3(n+2)L_{n+2}-20(n+1)F_{n+2}-10nF_{n+3}+10L_{n+4}-50\right\}$$

$$=F_{n+2}^{(2)}-\frac{1}{25}[10n(2F_{n+2}+F_{n+3})+10(2F_{n+2}-L_{n+4})+50]$$

$$=F_{n+2}^{(2)}-\frac{1}{5}[2nL_{n+3}-2(L_{n+4}-2F_{n+2})]-2=F_{n+2}^{(2)}+\frac{2}{5}(F_{n+5}-F_n-nL_{n+3})-2$$

$$=F_{n+2}^{(2)}-2F_{n+3}^{(1)}+\frac{2}{5}(F_{n+4}-F_n+3L_{n+3})-2=F_{n+2}^{(2)}-2F_{n+3}^{(1)}+2F_{n+4}-2. \quad \square$$

Proof of Proposition 7: From (2.7), we can write

$$B_n=\sum_{i=0}^{n}\binom{n}{i}F_i^{(2)}=\frac{1}{5}\left[\sum_{i=0}^{n}\binom{n}{i}iL_i^{(1)}-3\sum_{i=0}^{n}\binom{n}{i}F_i^{(1)}-\sum_{i=0}^{n}\binom{n}{i}F_i\right]. \tag{5.7}$$

Now, from [4, (3.5), (3.10), (3.3)], we have

$$\sum_{i=0}^{n}\binom{n}{i}iL_i^{(1)}=\sum_{i=0}^{n}\binom{n}{i}i^2F_i=nF_{2n-1}+n(n-1)F_{2n-2}, \tag{5.8}$$

$$\sum_{i=0}^{n}\binom{n}{i}F_i^{(1)}=F_{2n-1}^{(1)}/2=\frac{1}{10}[(2n-1)L_{2n-1}-F_{2n-1}], \tag{5.9}$$

$$\sum_{i=0}^{n}\binom{n}{i}F_i=F_{2n}, \tag{5.10}$$

respectively. Therefore, from (5.8)-(5.10) and (1.8), (5.7) can be rewritten as

$$B_n = \frac{1}{5}\left[nF_{2n-1} + n(n-1)F_{2n-2} - \frac{3(2n-1)L_{2n-1} - 3F_{2n-1}}{10} - F_{2n}\right]$$
$$= \frac{1}{50}\left[10nF_{2n-1} + 10n^2F_{2n-2} - 10nF_{2n-2} - 6nL_{2n-1} + 3L_{2n-1} + 3F_{2n-1} - 10F_{2n}\right]$$
$$= \frac{1}{25}\left[5n^2F_{2n-2} + n(5F_{2n-1} - 5F_{2n-2} - 3L_{2n-1}) - 2F_{2n}\right]$$
$$= \frac{1}{25}\left[5n^2F_{2n-2} + n(F_{2n-7} - 3F_{2n}) - 2F_{2n}\right]. \quad \square$$

6. FURTHER RESEARCH

The first and the second derivatives of polynomials (1.6) and (1.7) have been considered in [4] and in this paper, respectively. More particularly, several properties of the sequences of integers obtainable by taking the above mentioned derivatives at $x = 1$ have been investigated.

The generalization to the analogous sequences $\{F_n^{(k)}\}$ and $\{L_n^{(k)}\}$, defined as

$$F_n^{(k)} = \left[\frac{d^k}{dx^k}U_n(x)\right]_{x=1} = \sum_{j=0}^{\lfloor (n-k-1)/2 \rfloor}\left[\binom{n-1-j}{j}\prod_{i=1}^{k}(n-i-2j)\right] \quad (n \ge 1) \tag{6.1}$$

and

$$L_n^{(k)} = \left[\frac{d^k}{dx^k}V_n(x)\right]_{x=1} = \sum_{j=0}^{\lfloor (n-k)/2 \rfloor}\left[\frac{n}{n-j}\binom{n-j}{j}\prod_{i=1}^{k}(n-i+1-2j)\right] \quad (n \ge 1) \tag{6.2}$$

(with $F_0^{(k)} = 0$ for $k \ge 0$ and $L_0^{(k)} = 0$ for $k \ge 1$), seems to be very interesting and will be the goal of a future work. In this section we confine ourselves to offering some conjectures about the properties of these sequences.

Conjecture 1: $L_n^{(k)} = nF_n^{(k-1)}$.

Conjecture 2: $L_n^{(k)} = (n-k+1)L_n^{(k-1)} - 2\left(L_{n-1}^{(k)} + F_{n-1}^{(k-1)}\right)$.

Conjecture 3: $F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + kF_{n-1}^{(k-1)}$.

Conjecture 4: $L_n^{(k)} = L_{n-1}^{(k)} + L_{n-2}^{(k)} + kL_{n-1}^{(k-1)}$.

Conjecture 5: $F_{n-1}^{(k)} + F_{n+1}^{(k)} = L_n^{(k)}$.

Conjecture 6: $F_n^{(k)} \equiv L_n^{(k)} \equiv 0 \pmod 2$ for $k \ge 2$.

Conjecture 7: $L_n^{(k)} \equiv 0 \pmod n$ for $k \ge 1$.

Moreover, we leave to the reader the proof of the following:

$$L_n^{(n)} = L_n^{(n-1)} = n! \quad (n \ge 1), \tag{6.3}$$

$$L_n^{(n-2)} = \frac{n+1}{2(n-1)}n! \quad (n \ge 2), \tag{6.4}$$

$$L_n^{(n-3)} = \frac{n+5}{6(n-1)}n! \quad (n \ge 3), \tag{6.5}$$

$$L_n^{(n-4)} = \frac{n+10}{24(n-2)} n! \quad (n \geq 4). \tag{6.6}$$

Observe that (6.3)-(6.6) hold also for the minimum admissible value v of n, for which one has $L_v^{(0)} = L_v$. Analogous identities for $F_n^{(k)}$ can be stated whence the validity of Conjecture 1 can be checked. More generally, all the conjectures and results presented above can be checked against the numerical triangles shown in Figures 1 and 2, which have been obtained by (6.1) and (6.2), respectively. It must be noted that $F_n^{(k)} = 0$ for $k > n-1$, whereas $L_n^{(k)} = 0$ for $k > n$.

n \ k	0	1	2	3	4	5	6	7	8
0	0								
1	1	0							
2	1	1	0						
3	2	2	2	0					
4	3	5	6	6	0				
5	5	10	18	24	24	0			
6	8	20	44	84	120	120	0		
7	13	38	102	240	480	720	720	0	
8	21	71	222	630	1560	3240	5040	5040	0

Fig. 1. Triangle $F_n^{(k)}$ $(0 \leq n, k \leq 8)$

n \ k	0	1	2	3	4	5	6	7	8
0	2								
1	1	1							
2	3	2	2						
3	4	6	6	6					
4	7	12	20	24	24				
5	11	25	50	90	120	120			
6	18	48	120	264	504	720	720		
7	29	91	266	714	1680	3360	5040	5040	
8	47	168	568	1776	5040	12480	25920	40320	40320

Fig. 2. Triangle $L_n^{(k)}$ $(0 \leq n, k \leq 8)$

As indicated at the end of [4], the theory in this paper can be extended to cover Pell polynomials and numbers, and Pell-Lucas polynomials and numbers. In this case, we first replace x by $2x$ in (1.1) and (1.2), differentiate, and then put $x = 1$.

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