ON POINTS WHOSE COORDINATES ARE TERMS OF A LINEAR RECURRENCE*

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1. INTRODUCTION

Let $R = \{R_n\}_{n=0}^{\infty}$ be a second-order recurrent sequence (generalized Fibonacci sequence) of integers defined by

$$R_n = AR_{n-1} - BR_{n-2} \quad (\text{for } n > 1),$$

where the initial terms are $R_0 = 0$, $R_1 = 1$, and A and B are fixed nonzero integers. Let α and β be the roots of the characteristic polynomial $x^2 - Ax + B$. We will assume that the discriminant $D = A^2 - 4B > 0$ and D is not a perfect square. From this, it follows that the sequence R is not degenerate, i.e., α / β is not a root of unity. In this case, α and β are two irrational real numbers and $|\alpha| \neq |\beta|$, so we can suppose that $|\alpha| > |\beta|$. Also, $0 < \beta$ iff $0 < A \cdot B$. And $0 < \beta < 1$ holds iff $0 < B(A - B - 1)$.

It is well known that the terms of R can be given by

$$R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{D}}. \tag{1}$$

Furthermore

$$\lim_{n \to \infty} \frac{R_{n+1}}{R_n} = \alpha \tag{2}$$

(see, e.g., [3] or [6]).

Limit (2) implies that α can be approximated by the rational numbers R_{n+1} / R_n. The second author, P. Kiss [5], proved that when $B = 1$ this approximation is good in the sense that

$$\left| \alpha - \frac{R_{n+1}}{R_n} \right| < \frac{1}{c \cdot R_n^2}$$

holds for some c and infinitely many n.

It was also proved in [5] that this inequality holds for infinitely many n only when $|B| = 1$.

In this paper the points $P_n = (R_n, R_{n+1})$ will be considered from a geometric point of view, as points on the Euclidean plane. G. E. Bergum [1] and A. F. Horadam [2] showed that the points $P_n = (x, y)$ lie on the conic section $Bx^2 - Axy + y^2 + eB^n = 0$, where $e = AR_0R_1 - BR_0^2 - R_1^2$, and

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the initial terms R_0 and R_1 are not necessarily 0 and 1. In their treatment of this equation, they showed that in the case $|B|=1$, when the conic is a hyperbola, the asymptotes of the hyperbola are the lines $y=\alpha x$ and $y=\beta x$. This corresponds to limit (2). For the Fibonacci sequence, when $A=1$ and $B=-1$, C. Kimberling [4] characterized those conics satisfied by infinitely many Fibonacci lattice points $(x,y)=(F_m,F_n)$.

In this paper we again investigate the geometric properties of P_n in both the two- and three-dimensional cases.

2. THE TWO-DIMENSIONAL CASE

Let us consider the points $P_n=(R_n,R_{n+1})$, $n=0,1,2,\ldots$, on the plane whose coordinates are consecutive elements of the sequence R. Then (2) shows that the slope of the vector OP_n tends to α. But it is not obvious that the points P_n approach the line $y=\alpha x$, as $n\to\infty$. The following theorem shows that this is the case, however.

Theorem 1: Let d_n denote the distance from the point $P_n=(R_n,R_{n+1})$ to the line $y=\alpha x$. Then $\lim_{n\to\infty}d_n=0$ if and only if $|\beta|<1$.

Proof: The distance d_{x_0,y_0} from a point (x_0,y_0) to the line $y=\alpha x$ is given by

$$d_{x_0,y_0}=\left|\frac{\alpha x_0-y_0}{\sqrt{\alpha^2+1}}\right|, \tag{3}$$

so, using (1), we have

$$d_n=\left|\frac{\alpha R_n-R_{n+1}}{\sqrt{\alpha^2+1}}\right|=\left|\frac{\dfrac{\alpha^{n+1}-\beta^n\alpha}{\alpha-\beta}-\dfrac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}}{\sqrt{\alpha^2+1}}\right|=\frac{|\beta|^n}{\sqrt{\alpha^2+1}} \tag{4}$$

from which the theorem follows.

Remark: $|\beta|<1$ holds when $|B+1|<|A|$.

This theorem implies that the points P_n converge to the line $y=\alpha x$ if $|\beta|<1$, but not necessarily that these lattice points P_n are the nearest (in the sense of Theorem 2) lattice points to $y=\alpha x$ in all cases. For, let $d_{x,y}$ denote the distance between the lattice point (x,y) and the line $y=\alpha x$, and let d_n be the distance mentioned in the theorem. We prove

Theorem 2: For integers u, v, denote by $d_{u,v}$ the distance from the lattice point (u,v) to the line $y=\alpha x$ and let d_n be the distance defined in Theorem 1. Then when $|B|=1$, there is no lattice point (x,y) such that $d_{x,y}\le d_n$ and $|x|<|R_n|$. Furthermore, for sufficiently large n, this holds if and only if $|B|=1$.

Proof: First suppose $|B|=1$. In this case, obviously, $|\beta|<1$ and α is irrational. Assume that for some n there is a lattice point (x,y) such that $d_{x,y}\le d_n$ and $|x|<|R_n|$. Then, by (3) and

(4), $|\alpha x - y| \le |\beta|^n$ follows. From this, using (1) and the fact that $|\alpha\beta| = |B| = 1$, we obtain the inequalities

$$\left|\alpha - \frac{y}{x}\right| \le \frac{|\beta|^n}{|x|} = \frac{1}{|\alpha|^n |x|} = \frac{|1-(\beta/\alpha)^n|}{\sqrt{D}\cdot|R_n x|} < \frac{|1-(\beta/\alpha)^n|}{\sqrt{D}\cdot x^2}. \tag{5}$$

In [5] it was proved that if $|B| = 1$, and p, q are integers such that $(p, q) = 1$ and

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{D}\cdot q^2}$$

then p/q has the form $p/q = R_{i+1}/R_i$ for some i. The proof also shows that (5) holds only if $x = R_i$ and $y = R_{i+1}$ for some i, if n is large enough. So $x = R_i$ is a term of the sequence R. The sequence R is a nondegenerate one with $D > 0$ and $|B| = 1$. So it can easily be seen that $|R_t|$, $|R_{t+1}|$, ..., is an increasing sequence if t is sufficiently large. Furthermore, by (4), $d_k > d_j$ if $k < j$. Thus, $i < n$ and $d_i > d_n$ follows, which contradicts $d_i = d_{x,y} \le d_n$. So the first part of the theorem is proved.

To complete the proof, we have to prove that if $|B| > 1$, then there are infinitely many pairs of lattice points (x, y) such that $d_{x,y} < d_n$ and $|x| < |R_n|$ for any sufficiently large n.

Suppose $|B| > 1$. If $|\beta| < 1$, then, by (4), $d_n \to \infty$ as $n \to \infty$, so there are lattice points (x, y) such that $d_{x,y} < d_n$ and $|x| < |R_n|$ for any sufficiently large n.

If $|\beta| = 1$, then d_n is a constant and there are infinitely many points (x, y) such that $d_{x,y} \le d_n$ and $|x| < |R_n|$ for some n, since $|R_n| \to \infty$ as $n \to \infty$.

If $|\beta| < 1$, then by (4) and $|B| > 1$, we have

$$\left|\alpha - \frac{R_{n+1}}{R_n}\right| = \frac{|\beta|^n}{|R_n|} = \frac{|B|^n |1-(\beta/\alpha)^n|}{\sqrt{D}\cdot R_n^2} > \frac{Q}{R_n^2} \tag{6}$$

for any fixed $Q > 0$ if n is sufficiently large. In this case, the roots α, β are irrational numbers since, if the roots of the polynomial $x^2 - Ax + B$ are rational, then they are integers; so $0 < |\beta| < 1$ would be impossible. It is known that if $r_k = y/x$ is a convergent of the continued fraction expansion of α, then

$$\left|\alpha - \frac{y}{x}\right| < \frac{1}{2|x|^2}. \tag{7}$$

Let y, and hence x, be large enough and let the index n be defined by $|R_{n-1}| \le |x| < |R_n|$. From (3), (4), (6), and (7), we obtain the inequalities

$$d_n > \frac{Q}{|R_n|\cdot\sqrt{\alpha^2+1}} \quad \text{and} \quad d_{x,y} < \frac{1}{2|x|\sqrt{\alpha^2+1}}.$$

But, by (1),

$$\frac{Q}{|R_n|} = \frac{Q}{|R_{n-1}\alpha|(1-(\beta/\alpha)^n)/(1-(\beta/\alpha)^{n-1})} > \frac{1}{2|R_{n-1}|} \geq \frac{1}{2|x|}$$

and so $d_{x,y} < d_n$ with $|x| < |R_n|$, which completes the proof of the theorem.

Lastly, we give equations that are satisfied by the lattice points (R_n, R_{n+1}).

Theorem 3: All lattice points $(x, y) = (R_n, R_{n+1})$ satisfy one of the equations

$$\textbf{\textit{(i)}}\ \ y = \alpha x + c(x)\cdot|x|^{\delta} \quad \text{or} \quad \textbf{\textit{(ii)}}\ \ y = \alpha x - c(x)\cdot|x|^{\delta},$$

where $\delta = \log|\beta|/\log|\alpha|$ and $c(x)$ is a function such that $\lim_{x\to\infty} c(x) = \sqrt{D}^{\delta}$.

Remark: This shows that the sequence of lattice points (R_n, R_{n+1}) tends to the line $y = \alpha x$ only if $\delta < 0$, i.e., iff $|\beta| < 1$, as proved in Theorem 1.

Proof: By (1), we have

$$R_{n+1} = \alpha\frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha\beta^n - \beta^{n+1}}{\alpha - \beta} = \alpha R_n + \beta^n \tag{8}$$

and

$$|R_n| = \frac{|\alpha|^n}{\sqrt{D}}(1-(\beta/\alpha)^n). \tag{9}$$

From (9), we have $n = \frac{\log|R_n| + \log\sqrt{D} - \varepsilon_n}{\log|\alpha|}$ where $\varepsilon_n = \log(1-(\beta/\alpha)^n)$ and $\varepsilon_n \to 0$ as $n\to\infty$ since $|\beta/\alpha| < 1$. This implies that

$$\beta^n = \pm\exp\left\{\frac{\log|\beta|\cdot\log|R_n|}{\log|\alpha|} + \frac{\log|\beta|\cdot\log\sqrt{D}}{\log|\alpha|} - \frac{\varepsilon_n\cdot\log|\beta|}{\log|\alpha|}\right\} = \pm|R_n|^{\delta}\cdot\sqrt{D}^{\delta_n} \tag{10}$$

where $\delta = \log|\beta|/\log|\alpha|$ and

$$\delta_n = \frac{\log|\beta|}{\log|\alpha|} - \frac{\varepsilon_n\cdot\log|\beta|}{\log\sqrt{D}\cdot\log|\alpha|} \to \delta \text{ as } n\to\infty, \tag{11}$$

since $\varepsilon_n \to 0$ as $n\to\infty$.

From (8), (10), and (11), the theorem follows.

Remark: The lattice points (R_n, R_{n+1}) safisfy ***(i)*** for every n if $\beta > 0$. If $\beta < 0$, then the lattice points satisfy alternately ***(i)*** and ***(ii)***.

3. THE THREE-DIMENSIONAL CASE

Now we consider the three-dimensional vectors (R_n, R_{n+1}, R_{n+2}). Since by (1),

$$\frac{R_{n+2}}{R_n} = \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} = \alpha^2\frac{(1-(\beta/\alpha)^{n+1})}{1-(\beta/\alpha)^n} \to \alpha^2, \text{ as } n\to\infty,$$

$R_{n+1}/R_n \to \alpha$, as $n \to \infty$, by (2), and

$$(R_n, R_{n+1}, R_{n+2}) = R_n\left(1, \frac{R_{n+1}}{R_n}, \frac{R_{n+2}}{R_n}\right).$$

That is, the direction of the vectors (R_n, R_{n+1}, R_{n+2}) tends to the direction of the vector $(1, \alpha, \alpha^2)$. However, the sequence of the lattice points $P_n = (R_n, R_{n+1}, R_{n+2})$ does not always tend to the line passing through the origin and parallel to the vector $(1, \alpha, \alpha^2)$. We will prove the analog of Theorem 1.

Theorem 4: let L be a line defined by $x = t, y = \alpha t, z = \alpha^2 t, t \in \mathbb{R}$. Furthermore, let d_n be the distance from the point (R_n, R_{n+1}, R_{n+2}), $n = 0, 1, 2, \ldots$, to the line L. Then $\lim_{n\to\infty} d_n = 0$ if and only if $|\beta| < 1$.

Proof: It is not difficult to show that the distance from the point (x_0, y_0, z_0) to the line L is

$$d_{x_0, y_0, z_0} = \sqrt{\frac{(x_0\alpha^2 - z_0)^2 + (x_0\alpha - y_0)^2 + (y_0\alpha^2 - z_0\alpha)^2}{1 + \alpha^2 + \alpha^4}}. \tag{12}$$

This notation is necessary for Theorem 5.

By (12) and (1), we have

$$\begin{aligned} d_n &= \sqrt{\frac{(\beta^{n+2} - \alpha^2\beta^n)^2 + (\beta^{n+1} - \alpha\beta^n)^2 + (\alpha\beta^{n+2} - \alpha^2\beta^{n+1})^2}{(\alpha - \beta)^2(1 + \alpha^2 + \alpha^4)}} \\ &= |\beta|^n \sqrt{\frac{(\alpha+\beta)^2 + 1 + (\alpha\beta)^2}{1 + \alpha^2 + \alpha^4}} = |\beta|^n \sqrt{\frac{A^2 + B^2 + 1}{1 + \alpha^2 + \alpha^4}}, \end{aligned} \tag{13}$$

where we have used $\alpha + \beta = A$ and $\alpha\beta = B$ since α and β are the zeros of the polynomial $x^2 - Ax + B$. From this, the theorem follows.

Theorem 2 can also be generalized to the three-dimensional case, i.e., to state that the lattice points (R_n, R_{n+1}, R_{n+2}) are the nearest lattice points to the line L iff $|B| = 1$.

Theorem 5: Let L be the line defined in Theorem 4. Let d_n and $d_{x,y,z}$ be the distances defined in Theorem 4 and its proof. Then, for sufficiently large n, there is no lattice point (x, y, z) such that $d_{x,y,z} \le d_n$ and $|x| < |R_n|$ if and only if $|B| = 1$.

Proof: Suppose $|B| = 1$. Then, since $0 < D = A^2 - 4B$, α is irrational because $A^2 \pm 4$ is not a perfect square.

Let (x, y, z) be a lattice point such that

$$d_{x,y,z} \le d_n \tag{14}$$

for some n and $|x|<|R_n|$. By Theorem 4 $d_{x,y,z}<\varepsilon$ follows for any $\varepsilon>0$ if n is sufficiently large. But then, by (12), $|x\alpha^2-z|, |x\alpha-y|$, and $|y\alpha^2-z\alpha|$ are sufficiently small. If $|x\alpha-y|$ is a small number then, since $\alpha^2=A\alpha-B$, $|x\alpha^2-z|=|Ax\alpha-(z+Bx)|$ can be small only if $z+Bx=Ay$, i.e., only if $z=Ay-Bx$. In this case

$$|x\alpha^2-z|=A\cdot|x\alpha-y| \text{ and } |y\alpha^2-z\alpha|=|(z-Ay)\alpha+By|=|B|\cdot|x\alpha-y|$$

are also small. Thus, from (12), (13), and (14),

$$d_{x,y,z}=\sqrt{\frac{A^2+B^2+1}{1+\alpha^2+\alpha^4}}\cdot|x\alpha-y|\le|\beta|^n\sqrt{\frac{A^2+B^2+1}{1+\alpha^2+\alpha^4}}$$

and so, using $|x|<|R_n|$ and $|\alpha\beta|=|B|=1$, we get

$$\left|\alpha-\frac{y}{x}\right|\le\frac{|\beta|^n}{|x|}=\frac{1}{|\alpha|^n|x|}=\frac{1-(\beta/\alpha)^n}{|R_n|\cdot\sqrt{D}\cdot|x|}<\frac{1-(\beta/\alpha)^n}{\sqrt{D}\cdot|x|^2}.$$

From this, as above, we obtain $x=R_i, y=R_{i+1}$, and $z=Ay-Bx=R_{i+2}$ for some natural number i, if n is sufficiently large. Thus, $d_{x,y,z}=d_i$. But by (13), $d_k<d_n$ only if $k>n$, so $i\ge n$ and $|x|=|R_i|\ge|R_n|$, which contradicts the assumption $|x|<|R_n|$, since the sequence $|R_n|$ is ultimately increasing.

To complete the proof, we have to show that in the case $|\beta|<1$ there are infinitely many lattice points (x,y,z) for which $d_{x,y,z}\le d_n$ and $|x|<|R_n|$ for some n. Such points trivially exist by (13), when $|\beta|>1$ or when $|\beta|=1$, so we can suppose that $|\beta|<1$.

Suppose $|B|>1$ and $|\beta|<1$. In this case α is irrational. Let $r=y/x$ be a convergent of the continued fraction expansion of α and let z be an integer defined by $z=Ay-Bx$. Then, by the elementary properties of continued fraction expansions of irrational numbers, using also the fact that $\alpha^2=A\alpha-B$, we have

$$|x\alpha-y|=x\left|\alpha-\frac{y}{x}\right|<\frac{2}{2|x|},$$

$$|x\alpha^2-z|=|Ax\alpha-(z+Bx)|=|Ax\alpha-Ay|=|Ax|\cdot\left|\alpha-\frac{y}{x}\right|<\frac{|A||}{2|x|},$$

and

$$|y\alpha^2-z\alpha|=|(z-Ay)\alpha+By|=|Bx|\cdot\left|\alpha-\frac{y}{x}\right|<\frac{|B|}{2|x|}.$$

This, together with (12), implies the inequality

$$d_{x,y,z} < \frac{1}{2|x|} \cdot \sqrt{\frac{A^2+B^2+1}{1+\alpha^2+\alpha^4}} = \frac{c}{2|x|} \quad \left(\text{for } c = \sqrt{\frac{A^2+B^2+1}{1+\alpha^2+\alpha^4}} \right). \tag{15}$$

Let n be a natural number defined by $|R_{n-1}| \le |x| < |R_n|$, For this n, by (13) and (15), we have

$$\begin{aligned} d_n &= |\beta|^n \sqrt{\frac{A^2+b^2+1}{1+\alpha^2+\alpha^4}} = \frac{|B|^n}{|\alpha|^n} \cdot c \\ &= \frac{|B|^n}{|\alpha|^{n-1}} \cdot \frac{c}{|\alpha|} = \frac{1}{|R_{n-1}|} \cdot \frac{c(1-(\beta/\alpha)^{n-1})|B|^n}{|\alpha| \cdot \sqrt{D}} > \frac{c}{2|x|} > d_{x,y,z} \end{aligned}$$

if x and hence n is large enough, since $|B| > 1$. This shows that, for any lattice point (x, y, z) defined as above, there is an n such that $d_{x,y,z} < d_n$ and $|x| < |R_n|$. This completes the proof.

Lastly we prove the three-dimensional analog of Theorem 3.

Theorem 6: The coordinates of the lattice points $(x, y, z) = (R_n, R_{n+1}, R_{n+2})$ satisfy the system of equations

$$\begin{aligned} &x = t \\ &y = \alpha t + c(t)|t|^\delta \quad \text{or} \quad y = \alpha t - c(t)|t|^\delta \\ &z = \alpha^2 t + A \cdot c(t)|t|^\delta \quad \text{or} \quad z = \alpha^2 t - A \cdot c(t)|t|^\delta \end{aligned}$$

where $\delta = \log|\beta| / \log|\alpha|$ and $c(t)$ is a real-valued function for which $\lim_{t\to\infty} c(t) = \sqrt{D}^{\delta}$.

Proof: By (1), it can easily be shown that

$$R_{n+2} = \alpha^2 R_n + (\alpha+\beta)\beta^n = \alpha^2 R_n + A\beta^n. \tag{16}$$

From (8), (10), (11), and (16), the theorem follows.

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