EULERIAN NUMBERS ASSOCIATED WITH SEQUENCES OF POLYNOMIALS

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1. INTRODUCTION

The well-known Eulerian numbers may be defined either by their generating function or as the coefficients of $\binom{t+n-k}{n}$, $k=0,1,\ldots,n$, in the factorial expansion of t^n. During their long history they were extensively studied (and frequently rediscovered) especially with respect to their number-theoretic properties and their connection with certain combinatorial problems (see [2], [3], [18], [19], and references therein). In the last decades, several interesting extensions and modifications were considered along with related combinatorial, probabilistic, and statistical applications ([4]-[8], [10], [12], [13], [15]).

The present paper was motivated by the problem of providing a unified approach to the study of Eulerian-related numbers, which on one hand will be general enough to cover the majority of the known cases and give rise to new sequences of numbers, but on the other will show up the common mathematical properties of the quantities under investigation.

In Section 2 we consider the expansion of a polynomial $p_n(t)$ in a series of factorials of order n and introduce the notion of p_n-associated Eulerian numbers and polynomials. Explicit expressions, recurrence relations, generating functions, and connection to other types of numbers are discussed. In Section 3 we first indicate how well-known results can be directly deduced through the general formulation and in the sequel discuss some additional interesting special cases. Section 4 deals with several statistical and mathematical applications. Finally, in Section 6, we proceed with a further generalization through exponential generating function considerations. A brief study of the most important properties of the generalized quantities is also included.

2. THE p_n-ASSOCIATED EULERIAN NUMBERS AND POLYNOMIALS

Let $\{p_n(t), n=0,1,\ldots\}$ be a class of polynomials with the degree of $p_n(t)$ being n and $p_0(t)=1$. The coefficients $A_{n,k}$ of the expansion of $p_n(t)$ in a series of factorials of degree n, namely

$$p_n(t)=\sum_{k=0}^{n} A_{n,k}\binom{t+n-k}{n} \tag{2.1}$$

will be called the **p_n-associated Eulerian numbers**.

The respective polynomial

$$A_n(t)=\sum_{k=0}^{n} A_{n,k}t^k \tag{2.2}$$

will be referred to as the **p_n-associated Eulerian polynomial**.

In Proposition 2.1 we provide an expression for $A_{n,k}$ and $A_n(t)$ through the polynomials $p_n(t)$.

Proposition 2.1:

a. $A_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} p_n(k-j).$

b. $A_n(t) = (1-t)^{n+1} \sum_{j=0}^{\infty} p_n(j) t^j.$

Proof: Making use of expansion (2.1) for $t = k-j$ and interchanging the order of summation, we obtain

$$\sum_{j=0}^{k} (-1)^j \binom{n+1}{j} p_n(k-j) = \sum_{r=0}^{n} A_{n,r} \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} \binom{n+k-j-r}{n}.$$

By virtue of Cauchy's formula, we have

$$\sum_{j=0}^{k} (-1)^j \binom{n+1}{j} \binom{n+k-j-r}{n} = (-1)^{k-r} \sum_{j=0}^{k-r} \binom{n+1}{j} \binom{-n-1}{k-r-j} = (-1)^{k-r} \delta_{kr}$$

and the first part of the proposition follows immediately. Finally, substituting the explicit expression of $A_{n,k}$ in $A_n(t)$ yields

$$A_n(t) = \left\{ \sum_{j=0}^{\infty} (-1)^j \binom{n+1}{j} t^j \right\} \left\{ \sum_{j=0}^{\infty} p_n(j) t^j \right\} = (1-t)^{n+1} \sum_{j=0}^{\infty} p_n(j) t^j. \quad \square$$

It is worth mentioning that the numbers $A_{n,k}$ can be expressed through finite difference operators as follows: If E is the displacement operator, $\nabla = 1 - E^{-1}$ and

$$\underline{p_{n,k}}(t) = \begin{cases} p_n(t) & \text{if } 0 \le t \le k \\ 0 & \text{otherwise} \end{cases}$$

then

$$A_{n,k} = \left[\nabla^{n+1} \underline{E^k p_n}(t) \right]_{t=0}.$$

A lot of numbers used in Combinatorial Analysis can be defined as coefficients of the expansion of a polynomial in a series of factorials. Well-known cases are the (usual and noncentral) Stirling numbers of the second kind, the Lah numbers ([16], [18], [19], and references therein), and the Gould-Hopper numbers ([9], [14]). The author [17] stated some general results for the numbers $P_{n,k}$ appearing in the expansion of an arbitrary polynomial $p_n(t)$ in a series of factorials, i.e.,

$$p_n(t) = \sum_{k=0}^{n} P_{n,k} (t)_k. \tag{2.3}$$

The next proposition furnishes the connection between the two double sequences of numbers $A_{n,k}$ and $P_{n,k}$.

Proposition 2.2: The p_n-associated Eulerian numbers $A_{n,k}$ are related to the numbers $P_{n,k}$ defined by (2.3), by

$$A_{n,k}=\sum_{j=0}^{k}(-1)^{k-j}j!\binom{n-j}{k-j}P_{n,j}, \tag{2.4}$$

$$P_{n,k}=\frac{1}{k!}\sum_{j=0}^{k}\binom{n-j}{k-j}A_{n,j}. \tag{2.5}$$

Proof: Proposition 2.1b, by virtue of (2.3) yields

$$A_n(t)=(1-t)^{n+1}\sum_{j=0}^{n}P_{n,j}\sum_{i=j}^{\infty}(i)_j t^i=\sum_{j=0}^{n}j!P_{n,j}t^j(1-t)^{n-j},$$

and expanding $(1-t)^{n-j}$ we deduce that

$$A_n(t)=\sum_{j=0}^{n}j!P_{n,j}\sum_{k=j}^{n}\binom{n-j}{k-j}(-1)^{k-j}t^k=\sum_{k=0}^{n}\left\{\sum_{j=0}^{k}(-1)^{k-j}j!\binom{n-j}{k-j}P_{n,j}\right\}t^k.$$

Comparing the last expression to (2.2), we immediately derive equality (2.4). The truth of (2.5) can be easily verified by inverting relation (2.4). □

We are now going to prove a result referring to recurrence relations satisfied by the numbers $A_{n,k}$ when a certain recurrence holds true for the polynomials $p_n(t)$. More specifically, we have

Proposition 2.3: If there exists a relation of the form

$$p_{n+1}(t)=(\alpha_n t+\beta_n)p_n(t)+(\gamma_n t+\delta_n)p_{n-1}(t),\quad n\geq 1, \tag{2.6}$$

connecting three polynomials with consecutive indices, then the numbers $A_{n,k}$ satisfy the next recurrence relation

$$\begin{aligned}A_{n+1,k}&=(k\alpha_n+\beta_n)A_{n,k}+[\alpha_n(n-k+2)-\beta_n]A_{n,k-1}+(k\gamma_n+\delta_n)A_{n-1,k}\\&\quad+[\gamma_n(n-2k+2)-2\delta_n]A_{n-1,k-1}+[-\gamma_n(n-k+2)+\delta_n]A_{n-1,k-2},\quad n\geq 1.\end{aligned} \tag{2.7}$$

Proof: Employing Proposition 2.1a and replacing $p_{n+1}(k-j)$ by virtue of (2.6), we obtain

$$\begin{aligned}A_{n+1,k}&=(\alpha_n k+\beta_n)\sum_{j=0}^{k}(-1)^j\binom{n+2}{j}p_n(k-j)-\alpha_n\sum_{j=1}^{k}(-1)^j j\binom{n+2}{j}p_n(k-j)\\&\quad+(\gamma_n k+\delta_n)\sum_{j=0}^{k}(-1)^j\binom{n+2}{j}p_{n-1}(k-j)-\gamma_n\sum_{j=1}^{k}(-1)^j j\binom{n+2}{j}p_{n-1}(k-j).\end{aligned}$$

Recurrence (2.7) is easily deduced by introducing the expressions

$$\binom{n+2}{j}=\binom{n+1}{j}+\binom{n+1}{j-1},\qquad j\binom{n+2}{j}=(n+2)\binom{n+1}{j-1},$$

$$\binom{n+2}{j}=\binom{n}{j}+2\binom{n}{j-1}+\binom{n}{j-2},\quad j\binom{n+2}{j}=(n+2)\left\{\binom{n}{j-1}+\binom{n}{j-2}\right\}$$

in the four summands appearing above, and making repeated use of Proposition 2.1a. □

It is worth mentioning that, in the special case $\gamma_n = \delta_n = 0$ [i.e., when the polynomials $p_n(t)$ have real roots], the resulting numbers $A_{n,k}$ consistitute a triangular array of numbers.

In the remainder of this section we shall establish a connection between the exponential generating function (egf) of the polynomials $p_n(t)$ and the respective egf of the p_n-associated Eulerian polynomials. The basic assumption made here is that the egf

$$P(t,u) = \sum_{n=0}^{\infty} p_n(t)\frac{u^n}{n!}$$

of the sequence of polynomials $p_n(t)$, $n = 0, 1, \ldots$, can be expressed in the form

$$P(t,u) = g(u)\exp[t(F(u)-F(0))] \tag{2.8}$$

with $g(0) = 1$. This setting is general enough to include a lot of important special cases with diverse applications to combinatorics, physics, and mathematical analysis itself, as will be indicated in the next section. We mention here in brief that the special case $g(u) = 1$ leads to the well-known **exponential Bell polynomials** which have been studied in great detail (see [1], [18], [19]).

Proposition 2.4: If the polynomials $p_n(t)$, $n = 0, 1, \ldots$, have egf of the form (2.8), then the egf of the p_n-associated Eulerian polynomials

$$A(t,u) = \sum_{n=0}^{\infty} A_n(t)\frac{u^n}{n!}$$

is given by

$$A(t,u) = g((1-t)u)\frac{1-t}{1-tf((1-t)u)} \tag{2.9}$$

where $f(u) = \exp[F(u)-F(0)]$.

Proof: By virtue of Proposition 2.1b, we find that

$$A(t,u) = (1-t)\sum_{n=0}^{\infty}\left\{\sum_{j=0}^{\infty} p_n(j)t^j\right\}\frac{[(1-t)u]^n}{n!} = (1-t)\sum_{j=0}^{\infty} P(j,(1-t)u)t^j$$

and on making use of (2.8) we easily deduce the desired expression (2.9). □

It is easily seen that $A(t,u)$ is the double generating function of the numbers $A_{n,k}$ and writing

$$A(t,u) = \sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty} A_{n,k}\frac{u^n}{n!}\right\}t^k$$

we conclude that the (single) egf of the numbers $A_{n,k}, n = k, k+1, \ldots$, may be obtained by computing the coefficients of t^k in the power series expansion of $A(t,u)$ with respect to t. This is, in general, a difficult task.

3. SPECIAL CASES

In this section we shall treat some important special cases of p_n-associated Eulerian numbers, obtained by making certain choices of the polynomials $p_n(t)$.

a. If $p_n(t)=(t+r)^n$, then the expansion formula (2.1) indicates that $A_{n,k}$ are the **cumulative numbers** used by Dwyer ([12], [13]) to express the ordinary moments of a frequency distribution in terms of the cumulative totals. Now we have

$$P(t,u)=e^{(t+r)u},\ F(u)=u,\ f(u)=e^u,\ g(u)=e^{ru},\ \alpha_n=1,\ \beta_n=r,\ \gamma_n=0,\ \delta_n=0,$$

and applying Propositions 2.1-2.4, we deduce that

$$A_{n,k}=\sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k+r-j)^n \text{ (see Dwyer [12], Theorem III, p. 292)},$$

$$A_n(t)=(1-t)^{n+1}\sum_{j=0}^{\infty}(j+r)^n t^j,$$

$$A_{n,k}=\sum_{j=0}^{k}(-1)^{k-j}j!\binom{n-j}{k-j}S_{-r}(n,j),\quad S_{-r}(n,k)=\frac{1}{k!}\sum_{j=0}^{k}\binom{n-j}{k-j}A_{n,j},$$

where $S_{-r}(n,k)$ are the non-central Stirling numbers of the second kind (see [16]),

$$A_{n+1,k}=(k+r)A_{n,k}+(n-k+2-r)A_{n,k-1} \text{ (Dwyer [12], p. 294)},$$

$$A(t,u)=\exp[r(1-t)u]\frac{1-t}{1-t\exp[(1-t)u]}.$$

We mention here that the corresponding p_n-associated Eulerian polynomials are closely related to the quantities $H_n(r|t)$, which were studied in detail by Carlitz [2]. Note also that, for $r=0$, the numbers $A_{n,k}$ coincide with the usual Eulerian numbers (see [2], [18], [19])

$$A_{n,k}=\sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k-j)^n \tag{3.1}$$

while $S_0(n,k)=S(n,k)$ are the Stirling numbers of the second kind.

b. If $p_n(t)=(st+r)_n$, then Proposition 2.1a yields

$$A_{n,k}=\sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(s(k-j)+r)_n=n!\,A(n,k,s,r)$$

where $A(n,k,s,r)$ are the composition numbers. These numbers, as pointed out in [7], have many applications in combinatorics and statistics. It is obvious that to comply with our general setting, we must take

$$P(t,u)=(1+u)^{st+r},\ F(u)=s\log(1+u),\ f(u)=(1+u)^s,\ g(u)=(1+u)^r,$$
$$\alpha_n=s,\ \beta_n=r-n,\ \gamma_n=\delta_n=0,$$

and applying Propositions 2.1-2.4 we may derive the explicit expressions, recurrence relations, and egf of $A_{n,k}$ given by Charalambides [7]. Note that the numbers $P_{n,k} = G(n, k; s, r)$ of Proposition 2.2 which appear in the expansion

$$(st+r)_n = \sum_{k=0}^{n} G(n, k; s, r)(t)_k$$

are the so-called Gould-Hopper numbers (see [9], [14]). Note also that the limit $\lim_{s\to\pm\infty} s^{-n} A_{n,k}$ yields the Dwyer numbers mentioned in **a**. We finally mention that the special case $r = 0$ corresponds to the numbers

$$A_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (s(k-j))_n = s^n A_{n,k}(s^{-1}) \tag{3.2}$$

where $A_{n,k}(\cdot)$ are the polynomials studied by Carlitz ([4], Ch. 7). As Carlitz, Roselle, & Scoville [5] pointed out, for $s < 0$, the number of ordered sets $(i_1, i_2, \ldots, i_n)$ with $i_j \in \{1, 2, \ldots, |s|\}$ and showing exactly k increases between adjacent elements, is equal to

$$|A_{n,k}|/n! = (-1)^n A_{n,k} / n!.$$

c. The author [17], motivated by the problem of providing explicit expressions for the distribution of two-sample sums from Poisson and binomial distributions, one of which is left-truncated, introduced the $r-q$ polynomials

$$r_n(t; s, r) = \frac{d^n}{dx^n}\left[x^{st} e^{-r(x-1)}\right]_{x=1}, \quad q_n(t; r) = \frac{d^n}{dx^n}\left[x^{r} e^{-t(x-1)}\right]_{x=1}.$$

Both sequences of polynomials comply with the restrictions set in the general context and give rise to two double sequences of numbers which, to our knowledge have not appeared in the literature yet. More specifically, we have

(i) The polynomials $r_n(t; s, r)$ satisfy the recurrence

$$r_{n+1}(t; s, r) = (r + st - n) r_n(t; s, r) + rn r_{n-1}(t; s, r), \quad n \ge 1,$$
$$r_0(t; s, r) = 1, \quad r_1(t; s, r) = st + r$$

and have egf

$$r(t, u; s, r) = \sum_{n=0}^{\infty} r_n(t; s, r) \frac{u^n}{n!} = (1+u)^{st} e^{ru}.$$

Therefore,

$$F(u) = s\log(1+u), \ f(u) = (1+u)^s, \ g(u) = e^{ru}, \ \alpha_n = s, \ \beta_n = r-n, \ \gamma_n = 0, \ \delta_n = rn,$$

and applying Propositions 2.2-2.4 for the numbers defined by the expansion

$$r_n(t; s, r) = \sum_{k=0}^{n} A_{n,k} \binom{t+n-k}{n},$$

we conclude that

$$A_{n+1,k} = (sk + r - n) A_{n,k} + [s(n-k+2) + n - r] A_{n,k-1} + rn[A_{n-1,k} - 2A_{n-1,k-1} + A_{n-1,k-2}], \quad n \ge 1,$$
$$A_{00} = 1, \ A_{10} = r, \ A_{11} = s - r,$$

$$A(t,u)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A_{n,k}t^k\frac{u^n}{n!}=e^{r(1-t)u}\frac{1-t}{1-t[1+(1-t)u]^s},$$
$$A_{n,k}=\sum_{j=0}^{k}(-1)^{k-j}j!\binom{n-j}{k-j}R(n,j;s,r),$$

where $R(n,k;s,r)$ are the numbers appearing in the convolution of two-sample sums from a binomial and a zero-truncated Poisson distribution (see [17]). Note that the numbers $A_{n,k}$ defined above give, in particular for $r=0$, the quantities (3.2).

(ii) For the polynomials $q_n(t;r)$, we have

$$q_{n+1}(t;r)=(r+t-n)q_n(t;r)+ntq_{n-1}(t),\quad n\geq 1,$$
$$q_0(t;r)=1,\ q_1(t;r)=t+r,$$
$$q(t,u;r)=\sum_{n=0}^{\infty}q_n(t;r)\frac{u^n}{n!}=(1+u)^r e^{tu}.$$

Hence,

$$F(u)=u,\ f(u)=e^u,\ g(u)=(1+u)^r,\ \alpha_n=1,\ \beta_n=r-n,\ \gamma_n=n,\ \delta_n=0,$$

and applying Propositions 2.2-2.4 for the numbers defined by the expansion

$$q_n(t;r)=\sum_{k=0}^{n}A_{n,k}\binom{t+n-k}{n}$$

we obtain

$$\begin{aligned}A_{n+1,k}&=(r-n+k)A_{n,k}+(2n-k-r+2)A_{n,k-1}\\&\quad+n\{kA_{n-1,k}+(n-2k+2)A_{n-1,k-1}-(n-k+2)A_{n-1,k-2}\},\quad n\geq 1,\\A_{00}&=1,\ A_{10}=r,\ A_{11}=1-r,\end{aligned}$$

$$A(t,u)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A_{n,k}t^k\frac{u^n}{n!}=[1+(1-t)u]^r\frac{1-t}{1-t\exp[(1-t)u]},$$
$$A_{n,k}=\sum_{j=0}^{k}(-1)^{k-j}j!\binom{n-j}{k-j}Q(n,j;r),\tag{3.3}$$

where

$$Q(n,k;r)=\sum_{j=k}^{n}\binom{n}{j}(r)_{n-j}S(j,k)$$

are the numbers appearing in the convolution of two-sample sums from a Poisson and a zero-truncated binomial distribution (see [17]). As can easily be verified from egf (3.3), the special case $r=0$ yields the usual Eulerian numbers (3.1).

d. The **Hermite Polynomials**

$$H_n(t)=(-1)^n e^{t^2}\frac{d^n e^{t^2}}{dt^n}$$

satisfy the recurrence

$$\begin{aligned}&H_{n+1}(t)=2tH_n(t)-2nH_{n-1}(t),\quad n\geq 1,\\&H_0(t)=1,\ H_1(t)=2t,\end{aligned}$$

while their egf is

$$H(t,u)=\sum_{n=0}^{\infty}H_n(t)\frac{u^n}{n!}=e^{-u^2+2ut}.$$

Thus,

$$F(u)=2u,\ f(u)=e^{2u},\ g(u)=e^{-u^2},\ \alpha_n=2,\ \beta_n=\gamma_n=0,\ \delta_n=-2n,$$

and the next results for the **Hermite-associated Eulerian numbers** $A_{n,k}$ are immediate consequences of Propositions 2.1-2.4:

$$A_{n,k}=\sum_{j=0}^{k}(-1)^j\binom{n+1}{j}H_n(k-j),$$

$$H_n(t)=\sum_{k=0}^{n}P_{n,k}(t)_k,\ \text{ where } P_{n,k}=\frac{1}{k!}\sum_{j=0}^{k}\binom{n-j}{k-j}A_{n,j},$$

$$\begin{aligned}&A_{n+1,k}=2kA_{n,k}+2(n-k+2)A_{n,k-1}-2n\{A_{n-1,k}-2A_{n-1,k-1}+A_{n-1,k-2}\},\quad n\geq 1,\\&A_{00}=1,\ A_{10}=0,\ A_{11}=2,\end{aligned}\tag{3.4}$$

$$A(t,u)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A_{n,k}t^k\frac{u^n}{n!}=\exp[-(1-t)^2u^2]\frac{1-t}{1-t\exp[2(1-t)u]}.$$

e. Another important class of polynomials encountered in several applications, especially in mathematical physics, consists of the (generalized) **Laguerre polynomials** $L_n^{(p)}(t)$, defined by

$$L_n^{(p)}(t)=\frac{1}{n!}e^t t^{-p}\frac{d^n}{dt^n}[e^{-t}t^{n+p}],\quad n=0,1,\ldots,p>-1.$$

Considering the polynomials

$$L_n(t;p)=n!\,L_n^{(p)}(t)=e^t t^{-p}\frac{d^n}{dt^n}[e^{-t}t^{n+p}],\quad n\geq 0,\ p>-1,$$

we get [making use of the respective results on $L_n^{(p)}(t)$]

$$\begin{aligned}&L_{n+1}(t;p)=(2n+p+1-t)L_n(t;p)-n(n+p)L_{n-1}(t;p),\quad n\geq 1,\\&L_0(t;p)=1,\ L_1(t;p)=-t+p+1,\\&L(t,u)=\sum_{n=0}^{\infty}L_n(t;p)\frac{u^n}{n!}=\sum_{n=0}^{\infty}L_n^{(p)}(t)u^n=(1-u)^{-p-1}\exp\left\{-\frac{tu}{1-u}\right\},\ |t|<1.\end{aligned}$$

Therefore,

$$\begin{aligned}&F(u)=\frac{u}{u-1},\ f(u)=\exp\left\{\frac{u}{u-1}\right\},\ g(u)=(1-u)^{-p-1},\\&\alpha_n=-1,\ \beta_n=2n+p+1,\ \gamma_n=0,\ \delta_n=-n(n+p),\end{aligned}$$

and applying Propositions 2.1-2.4, we deduce the following properties of the **Laguerre-associated Eulerian numbers** $A_{n,k}$:

$$A_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} L_n(k-j;\, p),$$

$$L_n(t) = \sum_{k=0}^{n} P_{n,k}(t)_k, \quad \text{where } P_{n,k} = \frac{1}{k!}\sum_{j=0}^{k}\binom{n-j}{k-j}A_{n,j},$$

$$A_{n+1,k} = (2n+p+1-k)A_{n,k} - (3n-k+p+3)A_{n,k-1}$$
$$-n(n+p)\{A_{n-1,k} - 2A_{n-1,k-1} + A_{n-1,k-2}\}, \quad n \ge 1,$$

$$A_{00} = 1, \ A_{10} = p+1, \ A_{11} = -p-2,$$

$$A(t,u) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{n,k} t^k \frac{u^n}{n!} = \frac{1}{[1-(1-t)u]^{p+1}} \cdot \frac{1-t}{1-t\exp\left\{\frac{(1-t)u}{(1-t)u-1}\right\}}.$$

4. APPLICATIONS

In the present section we consider a number of applications involving the p_n-associated Eulerian numbers and polynomials.

The first application refers to the computation of the mean value of polynomial functions of logarithmic random variables. More specifically, consider a random variable X with the logarithmic series distribution

$$P[X = x] = -\frac{1}{\log(1-\theta)} \cdot \frac{\theta^x}{x}, \quad x = 1, 2, \ldots, \quad 0 < \theta < 1,$$

and let $p_n(\cdot)$ be a polynomial of degree n. Then

$$v_n = E[Xp_n(X)] = c\left\{\sum_{x=0}^{\infty} p_n(x)\theta^x - p_n(0)\right\}, \quad c = -1/\log(1-\theta),$$

and employing the p_n-associated Eulerian polynomials $A_n(t)$, we may write, by virtue of Proposition 2.1b,

$$v_n = c\{(1-\theta)^{-n-1}A_n(\theta) - p_n(0)\}. \tag{4.1}$$

Formula (4.1) is useful for the derivation of recurrence relations for the quantities v_n [mean value of an $(n+1)$-degree polynomial with no constant term] by making use of the respective recurrence relations of the Eulerian polynomials $A_n(\theta)$. Note also that, under the assumptions made in Proposition 2.4, the egf of v_n, $n = 0, 1, \ldots$, is given by

$$\sum_{n=0}^{\infty} v_n \frac{u^n}{n!} = cg(u)\frac{\theta f(u)}{1-\theta f(u)}.$$

We mention in particular (see Section 3, cases a and b) that

$$\sum_{n=0}^{\infty} E[X(X+r)^n]\frac{u^n}{n!} = c\cdot\frac{\theta e^{(r+1)u}}{1-\theta e^u};$$

$$\sum_{n=0}^{\infty} E[X(sX+r)_n]\frac{u^n}{n!} = c\cdot\frac{\theta(1+u)^{r+s}}{1-\theta(1+u)^s}.$$

The second statistical application of the p_n-associated Eulerian numbers is in the computation of the polynomial mean of a frequency distribution with the use of cumulative totals. This method was used by Dwyer [12], [13] for the computation of the moments and by Charalambides [7] for the factorial moments. The main advantage of the method lies in the fact that the many multiplications involved in the usual computation process are replaced by additions. Since the generalization presented here is rather straightforward, we omit the details and state only the results. Let f_x denote a frequency distribution and

$$Cf_x = \sum_{j\geq x} f_j, \quad C^{m+1}f_x = C(C^m f_x), \quad m = 1, 2, \ldots,$$

the **successive frequency cumulations**. Then, employing Dwyer's successive cumulation theorem, we may easily deduce that for any polynomial $p_n(\cdot)$,

$$E[p_n(X)] = \sum_{j\geq 0} p_n(j) f_j = \sum_{k=0}^{n} A_{n,k} C^{n+1} f_k,$$

where $A_{n,k}$ are the Eulerian numbers corresponding to $p_n(\cdot)$.

As a last application, we consider the problem of evaluating the sum of the values of a polynomial $p_n(\cdot)$ over the first $m+1$ nonnegative integers, namely, $S = \sum_{x=0}^{m} p_n(x)$. Because of (2.1) we may write

$$S = \sum_{x=0}^{m}\sum_{k=0}^{n} A_{n,k}\binom{x+n-k}{n} = \sum_{k=0}^{n} A_{n,k}\sum_{x=0}^{m}\binom{x+n-k}{n},$$

and since the inner sum equals $\binom{m+n-k+1}{n+1}$, it follows that

$$\sum_{x=0}^{m} p_n(x) = \sum_{k=0}^{n} A_{n,k}\binom{m+n-k+1}{n+1}. \tag{4.2}$$

Consider in particular the next two special cases:

(i) Let $p_n(t) = p_3(t) = (st)_3$. Then, by virtue of (3.2) (or employing the respective recurrence relation for $A_{n,k}$) we get $A_{30} = 1$, $A_{31} = (s)_3$, $A_{32} = 4(s+1)_3$, $A_{33} = (s+2)_3$, and (4.2) yields

$$\sum_{x=0}^{m} (sx)_3 = \binom{m+4}{4} + (s)_3\binom{m+3}{4} + 4(s+1)_3\binom{m+2}{4} + (s+2)_3\binom{m+1}{4}.$$

(ii) Let $p_n(t) = H_2(t)$ denote the Hermite polynomial of degree 2. Recurrence (3.4) yields $A_{20} = -2$, $A_{21} = 8$, $A_{22} = 2$, and, therefore,

$$\sum_{x=0}^{m} H_2(x) = -2\binom{m+3}{3} + 8\binom{m+2}{3} + 2\binom{m+1}{3} = \frac{2(m+1)}{3}\{2m^2 + m - 3\}.$$

5. THE GENERALIZED p_n-ASSOCIATED EULERIAN NUMERS AND POLYNOMIALS

Carlitz & Scoville [6] introduced the generalized Eulerian numbers in connection with the problem of enumerating (a, b)-sequences (generalized permutations). These numbers, which are also related to Janardan's [15] generalized Eulerian numbers (used for the statistical analysis of an interesting ecology model), were extensively studied by Charalambides [8]. Recently, Charalambides & Koutras [10] considered an alternative ecology model and introduced a double sequence of numbers that are asymptotically connected with the numbers of Carlitz & Scoville.

In the present section we provide a unified approach to generalizations of this kind, bringing into focus the common properties of them and supplying the means for further extensions.

Let $\{p_n(t),\ n = 0, 1, \ldots\}$ be a class of polynomials with egf given by (2.8). Then, the numbers $A_{n,k}(a, b)$ with egf

$$A(t, u; a, b) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{n,k}(a, b)t^k \frac{u^n}{n!} = g((1-t)u)f^a((1-t)u)\left\{\frac{1-t}{1-tf((1-t)u)}\right\}^{a+b} \tag{5.1}$$

will be called **generalized p_n-associated Eulerian numbers**. Similarly, the polynomials

$$A_n(t; a, b) = \sum_{k=0}^{n} A_{n,k}(a, b)t^k$$

will be named **generalized p_n-associated Eulerian polynomials**. It is evident that

$$A_{n,k}(0, 1) = A_{n,k}, \quad A_n(t; 0, 1) = A_n(t).$$

Proposition 5.1:

a. $$A_n(t; a, b) = (1-t)^{n+a+b}\sum_{j=0}^{\infty}\binom{a+b+j-1}{j}t^j p_n(a+j).$$

b. $$A_{n,k}(a, b) = \sum_{j=0}^{k}(-1)^j\binom{n+a+b}{j}\binom{a+b+k-j-1}{k-j}p_n(a+k-j).$$

Proof: Expanding the term $[1-tf((1-t)u)]^{-(a+b)}$ of (5.1) yields

$$A(t, u; a, b) = (1-t)^{a+b}\sum_{j=0}^{\infty}\binom{a+b+j-1}{j}t^j\{g((1-t)u)f^{a+j}((1-t)u)\},$$

and applying (2.8) on the extreme right term, we obtain

$$A(t, u; a, b) = (1-t)^{a+b}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\binom{a+b+j-1}{j}t^j p_n(a+j)\frac{[(1-t)u]^n}{n!}.$$

The first part of the proposition is readily established by interchanging the order of summation and considering the coefficient of $u^n/n!$ in the resulting power series. The second part follows immediately from ***a*** by expanding $(1-t)^{n+a+b}$ and performing the multiplication of the two series. □

We note that, in particular, for $p_n(t) = t^n$ and $p_n(t) = (st)_n$, the numbers appearing in [6], [8], [15], and [10], respectively are obtained.

Taking the limit as $t \to 1$ in (5.1), it follows that

$$\lim_{t\to 1} A(t, u; a, b) = [1 - f'(0)u]^{-a-b} = \sum_{n=0}^{\infty} \binom{a+b+n-1}{n} [f'(0)]^n u^n,$$

implying

$$A_n(1; a, b) = \sum_{k=0}^{n} A_{n,k}(a, b) = (a+b+n-1)_n [f'(0)]^n. \tag{5.2}$$

Hence,

$$f_n(x; a, b) = P[X = x] = \frac{A_{n,x}(a, b)}{(a+b+n-1)_n [f'(0)]^n}, \quad x = 0, 1, 2, \ldots, n, \tag{5.3}$$

defines a legitimate probability function (provided that the ratios are nonnegative for all $x = 0, 1, \ldots, n$) which will be called **generalized p_n-associated Eulerian distribution**. It is straightforward that the probability generating function of (5.3) can be expressed as

$$E[t^X] = \sum_{x=0}^{n} f(x; a, b) t^x = \frac{A_n(t; a, b)}{(a+b+n-1)_n [f'(0)]^n},$$

while the factorial moment generating function is expressed as

$$\sum_{r=0}^{n} E[(X)_r] \frac{t^r}{r!} = \frac{A_n(t+1; a, b)}{(a+b+n-1)_n [f'(0)]^n}.$$

The next proposition provides recurrence relations for the numbers $A_{n,k}(a, b)$ and is useful for tabulation purposes [we recall also formula (5.2), which can be employed as a convenient check].

Proposition 5.2: Under the assumption that (2.6) is true, the numbers $A_{n,k}(a, b)$ satisfy the next recurrence relation:

$$\begin{aligned} A_{n+1,k}(a, b) &= [(a+k)\alpha_n + \beta_n] A_{n,k}(a, b) + [\alpha_n(n+b-k+1) - \beta_n] A_{n,k-1}(a, b) \\ &\quad + [(a+k)\gamma_n + \delta_n] A_{n-1,k}(a, b) + [\gamma_n(n+b-2k-a+1) - 2\delta_n] A_{n-1,k-1}(a, b) \\ &\quad + [-\gamma_n(n+b-k+1) + \delta_n] A_{n-1,k-2}(a, b), \quad n \ge 1. \end{aligned} \tag{5.4}$$

Proof: It is not difficult to verify that the auxiliary functions

$$C_n(t; a, b) = \sum_{j=0}^{\infty} \binom{a+b+j-1}{j} t^{a+j} p_n(a+j) = t^a (1-t)^{-(n+a+b)} A_n(t; a, b)$$

satisfy the difference-differential equation

$$C_{n+1}(t; a, b) = t \frac{d}{dt} \{\alpha_n C_n(t; a, b) + \gamma_n C_{n-1}(t; a, b)\} + \beta_n C_n(t; a, b) + \delta_n C_{n-1}(t; a, b).$$

Replacing $C_n(t; a, b)$ in terms of $A_n(t; a, b)$, we obtain a difference-differential equation for $A_n(t; a, b)$, and (5.4) is finally obtained after some lengthy but rather straightforward calculations. We mention that a proof similar to the one used in Proposition 2.3 could also be established; however, it is much more complicated. □

In the remainder of this section we are going to state some interesting results for generalized p_n-associated Eulerian numbers whose generating polynomials have real roots, i.e.,

$$\begin{aligned} p_{n+1}(t) &= \prod_{k=0}^{n}(\alpha_k t+\beta_k) = (\alpha_n t+\beta_n)p_n(t), \quad n\geq 0, \\ p_0(t) &= 1. \end{aligned} \tag{5.5}$$

In this case we have:

1. The numbers $A_{n,k}(a,b)$ satisfy the **triangular** recurrence relation

$$A_{n+1,k}(a,b) = [(a+k)\alpha_n+\beta_n]A_{n,k}(a,b)+[\alpha_n(n+b-k+1)-\beta_n]A_{n,k-1}(a,b).$$

2. The probability function $f(x;a,b)$ and the respective factorial moments $u_{(r)}(n;a,b) = E[(X)_r]$, $r=0,1,\ldots,$ satisfy the recurrences

$$f_{n+1}(x;a,b) = \frac{(a+k)\alpha_n+\beta_n}{(a+b+n)f'(0)}f_n(x;a,b)+\frac{\alpha_n(n+b-k+1)-\beta_n}{(a+b+n)f'(0)}f_n(x-1;a,b);$$

$$\mu_{(r)}(n+1;a,b) = \frac{r[\alpha_n(n+b-r+1)-\beta_n]}{(a+b+n)f'(0)}\mu_{(r-1)}(n;a,b)+\frac{\alpha_n(a+b+n-r)}{(a+b+n)f'(0)}\mu_{(r)}(n;a,b).$$

3. If $a\alpha_n+\beta_n\neq 0$ for all n = 0, 1, ..., then the polynomials $A_n(t;a,b)$ have n distinct real nonpositive roots (an easy way to prove this is to verify first that

$$E_n(t;a,b) = (1-t)^{-(n+a+b)}t^{\beta_n/\alpha_n+a}A_n(t;a,b)$$

satisfies a difference-differential equation of the form

$$\alpha_n t^s\frac{d}{dt}E_n(t;a,b) = E_{n+1}(t;a,b)).$$

Hence:

(a) $A_{n,k}(a,b)$ is a strictly concave function of k;

(b) the distribution $\{f_n(x;a,b),\ x=0,1,\ldots,n\}$ is unimodal either with a peak or with a plateau of two points (see [11]);

(c) Any random variable X obeying (5.3) can be expressed as a sum of n independent zero-one random variables.

We recall that the generalized Eulerian numbers studied in [6], [8], [10], [15], along with their generalizations produced by the choices $p_n(t)=(t+r)^n$, $p_n(t)=(st+r)_n$ (see Section 3, cases a and b), own the properties 1-3 above, since they are generated by polynomials of the form (5.5).

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