CROSS-JUMP NUMBERS

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Consider any \( n \)-digit integer expressed in the base \( b \). Divide it into a right part of \( r \) digits and a left part of \( n-r \) digits. To the left part add a number \( L < b \) and to the right part add some \( R < b \). The addition is done modulo \( b \) and the "carry-over" is ignored. Transfer the left part to the right of the right part and we again get an \( n \)-digit number. Apply this same process (which we call "cross-jumping") to the new number. Iterating this several times, we can ask if we get the original number back, and, if so, what is the least number \( N \) of steps required? We prove that

\[
N = \frac{bn}{(b, L+R) \cdot (n,r)}
\]

where \((a, b)\) denotes the G.C.D. of two numbers \( a \) and \( b \). We first illustrate this by an example.

**Example:** We take \( b=10, n=8, r=2, L=4, R=2 \). Starting with the number 56240317, the iteration gives

\[
\begin{align*}
56240317 & \to 26051556 & \to 07175426 \\
19562407 & \to 58260519 & \to 28071758 \\
09195628 & \to 11582609 & \to 50280711 \\
2091950 & \to 01115820 & \to 13502801 \\
52200913 & \to 22011152 & \to 03135022 \\
15522003 & \to 54220115 & \to 24031354 \\
05155224 & \to 17542205 & \to 56240317
\end{align*}
\]

which gives back the original number in the 20th step.

Let us prove our claimed formula for \( N \). We denote the positions of the \( n \) digits from left to right by 1, 2, ..., \( n \), respectively. The positions change as \( a \to a+r \to a+2r \) ... for each \( a \leq n \), where \( + \) is addition modulo \( n \). For repetition of the original number, we should have some \( k > 0 \) so that \( a + kr = a \mod n \). Clearly then, \( k = n / (n,r) \) is the least such \( k \). The choice of \( k \) only ensures that the positions of the original digits are the same after every \( k \) steps. Now, for any \( m \leq k = n / (n,k) \), there is a corresponding \( a_0 \) such that \( a_0 + mr = n \). We have

\[
a_0 \to a_0 + r \to a_0 + (m-1)r = n - r \to a_0 + mr = n \to a_0 + (m+1)r \to a_0 + kr = a_0,
\]

where we have written \( L, R \) over an arrow to indicate an increase in the value of that digit by \( L \), \( R \), etc. Thus, we have an increment of \( L + R \) in the value of each digit for every \( k \) steps. For repetition of the original number, this increment should be a multiple of \( b \) and, therefore, \( N \) must be a multiple of \( k \) as well as of \( kb/(L+R) \). This gives \( N = L.C.M. \) of \( k \) and \( kb/(L+R) \), i.e.,

\[
N = \frac{bn}{(b, L+R) \cdot (n,r)}.
\]

In our example, \( N = 20 \).

\[\star\star\star\]