SEQUENCES OF CONSECUTIVE n-NIVEN NUMBERS*

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A Niven number [2] is a positive integer that is divisible by the sum of its digits. In 1982, Kennedy [3] showed that there do not exist sequences of more than 21 consecutive Niven numbers. In 1992, Cooper & Kennedy [1] improved this result by proving that there does not exist a sequence of more than 20 consecutive Niven numbers. They also proved that this bound is the best possible by producing an infinite family of sequences of 20 consecutive Niven numbers.

For any positive integer $n \geq 2$, define an n-Niven number to be a positive integer that is divisible by the sum of the digits in its base n expansion. This paper examines the maximal possible lengths of sequences of consecutive n-Niven numbers. The main result is given in the following theorem.

Main Theorem: For any $n \geq 2$, there does not exist a sequence of more than $2n$ consecutive n-Niven numbers.

Note that this result is entirely general and gives Cooper & Kennedy's result as a special case.

Fix any $n \geq 2$. The following notation will be used throughout this paper. A number written in base n will be subscripted with (n). For example, $12 = 22_{(5)}$. When a string of (nonboldface) variables is subscripted, it is assumed that each variable represents a single digit in the given base. For example, $ij_{(n)}$ represents a number that can be expressed in two base n digits, $0 \leq i, j < n$. (Note that $i = 0$ is allowed.) A boldface variable in such a string represents a (possibly empty) string of digits in the given base. For example, $\mathbf{a}22_{(5)}$ represents a number congruent to 12 modulo 25. Let $s(\mathbf{a})$ be the sum of the digits in the string $\mathbf{a}$.

Lemma 1: Suppose that

$$\mathbf{a}0_{(n)}, \mathbf{a}1_{(n)}, \ldots, \mathbf{a}(n-1)_{(n)}$$

is a sequence of n consecutive n-Niven numbers. Then n divides $s(\mathbf{a})$.

Proof: Let $s = s(\mathbf{a})$. The base n digit sums of the numbers $\mathbf{a}0_{(n)}, \ldots, \mathbf{a}(n-1)_{(n)}$ are s, $s+1$, ..., $s+n-1$. Exactly one of the digit sums is divisible by n. The corresponding n-Niven number must also be divisible by n and thus must be $\mathbf{a}0_{(n)}$. Hence, n divides the digit sum of $\mathbf{a}0_{(n)}$, i.e., n divides s.

Lemma 2: The $n+1$ consecutive numbers $\mathbf{a}00_{(n)}, \ldots, \mathbf{a}10_{(n)}$ are not all n-Niven numbers.

Proof: Suppose to the contrary. Since n-Niven numbers are by definition positive, $s = s(\mathbf{a}) > 0$. Further, by Lemma 1, n divides s. Thus, $n \leq s$. The base n digit sum of both $\mathbf{a}01_{(n)}$ and

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$\mathbf{a}10_{(n)}$ is $s+1$. Since $s+1$ divides each, it must divide their difference, $n-1$. So $s+1 \le n-1 < s$. Contradiction.

Lemma 3: If $i \ne n-1$ and $s(\mathbf{a})+i > 0$, then $\mathbf{a}i(n-1)_{(n)}$ and $\mathbf{a}(i+1)(n-2)_{(n)}$ are not both n-Niven numbers.

Proof: Let $s = s(\mathbf{a})$. The base n digit sum of both $\mathbf{a}i(n-1)_{(n)}$ and $\mathbf{a}(i+1)(n-2)_{(n)}$ is $s+i+n-1$. If both are n-Niven numbers, then $s+i+n-1$ must also divide their difference, $n-1$. But $s+i>0$ implies that $s+i+n-1 > n-1$. Contradiction.

Theorem 4: If $\mathbf{a}ij_{(n)}$ is the first term in a sequence of length at least $2n$ of consecutive n-Niven numbers, then $i = n-1$ and $j = 0$.

Proof: Let $\mathbf{a}ij_{(n)}$ be the first term in such a sequence. Let $s = s(\mathbf{a})$. Suppose $i \ne n-1$. Then

$$\mathbf{a}(i+1)0_{(n)}, \mathbf{a}(i+1)1_{(n)}, \ldots, \mathbf{a}(i+1)(n-1)_{(n)}$$

is a subsequence of consecutive n-Niven numbers. By Lemma 1, n divides $s+i+1$, and so $s+1>0$. Further, both $\mathbf{a}i(n-1)_{(n)}$ and $\mathbf{a}(i+1)(n-2)_{(n)}$ are n-Niven numbers. But this is impossible by Lemma 3.

Similarly, if $i = n-1$ and $j \ne 0$, then the sequence contains the subsequence

$$(\mathbf{a}+1)00_{(n)}, \ldots, (\mathbf{a}+1)10$$

which cannot all be n-Niven numbers by Lemma 2.

We now prove the Main Theorem as an easy corollary to Theorem 4.

Proof of the Main Theorem: Suppose $x_1, x_2, \ldots$ is a sequence of more than $2n$ consecutive n-Niven numbers. By Theorem 4, both x_1 and x_2 end in zero when written in base n. This is clearly impossible.

It is not known whether this bound is the best possible. The earlier results show that it is the best possible for $n = 10$ and computer calculations have verified that it is optimal for a number of other small values of n. A general proof, however, that applies to all values of n has yet to be found.

Conjecture 5: For each $n \ge 2$, there exists a sequence of consecutive n-Niven numbers of length $2n$.

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