THE FIBONACCI AND LUCAS TRIANGLES MODULO 2

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(*Submitted May 1992*)

1. INTRODUCTION

The Fibonacci and Lucas coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{F}} = \frac{F_n F_{n-1} \cdots F_1}{(F_k F_{k-1} \cdots F_1)(F_{n-k} F_{n-k-1} \cdots F_1)}, \text{ and } \begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{L}} = \frac{L_n L_{n-1} \cdots L_1}{(L_k L_{k-1} \cdots L_1)(L_{n-k} L_{n-k-1} \cdots L_1)}.$$

These coefficients have been studied by several authors, [2], [8], [14], and [18]. Using these definitions, what we call the Fibonacci and Lucas triangles are formed in the same way as Pascal's Triangle is formed from ordinary binomial coefficients, that is, the n^{th} row is $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ for $0 \le k \le n$. Other authors, e.g., [3], [10], have also constructed such triangles in various ways. The ordinary binomial coefficients modulo 2 and Pascal's Triangle modulo 2 have been studied extensively in [4], [5], [6], [7], [11], [17], [20], [22], [23], and [25]. Among problems of interest have been the determination of the parity of binomial coefficients, the number of odd coefficients in the n^{th} row of Pascal's Triangle, and the iterative structure of Pascal's Triangle modulo 2. We will extend these results to both the Fibonacci and Lucas coefficients modulo 2 in sections 2 and 3. In section 4 we also determine the relationship between the Fibonacci and Lucas coefficients.

Portions of these triangles, both the originals and their modulo 2 reductions, are shown below. Since the Lucas coefficients are not always integers, the symbol α will be used to denote those coefficients, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{\mathcal{L}}$, that have a higher power of 2 in the denominator than in the numerator.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Pascal's Triangle

1
1 1
1 0 1
1 1 1 1
1 0 0 0 1
1 1 0 0 1 1
1 0 1 0 1 0 1
1 1 1 1 1 1 1 1
1 0 0 0 0 0 0 0 1

Pascal's Triangle Modulo 2

1
1 1
1 1 1
1 2 2 1
1 3 6 3 1
1 5 15 15 5 1
1 8 40 60 40 8 1
1 13 104 260 260 104 13 1
1 21 273 1092 1820 1092 273 21 1

Fibonacci Triangle

1
1 1
1 1 1
1 0 0 1
1 1 0 1 1
1 1 1 1 1 1
1 0 0 0 0 0 1
1 1 0 0 0 0 1 1
1 1 1 0 0 0 1 1 1

Fibonacci Triangle Modulo 2

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 3 \quad 1 \\
1 \quad 4 \quad 4 \quad 1 \\
1 \quad 7 \quad \frac{28}{3} \quad 7 \quad 1 \\
1 \quad 11 \quad \frac{77}{3} \quad \frac{77}{3} \quad 11 \quad 1 \\
1 \quad 18 \quad 66 \quad \frac{231}{2} \quad 66 \quad 18 \quad 1 \\
1 \quad 29 \quad 174 \quad \frac{957}{2} \quad \frac{957}{2} \quad 174 \quad 29 \quad 1 \\
1 \quad 47 \quad \frac{1363}{3} \quad \frac{4089}{2} \quad \frac{44979}{14} \quad \frac{4089}{2} \quad \frac{1363}{3} \quad 47 \quad 1
\end{array}$$

Lucas Triangle

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1 \quad 1 \\
1 \quad 0 \quad 0 \quad 1 \\
1 \quad 1 \quad 0 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\
1 \quad 0 \quad 0 \quad \alpha \quad 0 \quad 0 \quad 1 \\
1 \quad 1 \quad 0 \quad \alpha \quad \alpha \quad 0 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad \alpha \quad \alpha \quad \alpha \quad 1 \quad 1 \quad 1
\end{array}$$

Lucas Triangle Modulo 2

We will need the following information about regularly divisible sequences, generalized bases, and a generalized form of Kummer's Theorem.

Divisibility questions about sequences, such as which terms are divisible by a given prime, have been investigated by several authors, e.g., [9], [13], [15]. A sequence $\{u_n\}$ is said to be strongly divisible provided

$$\gcd(u_m, u_n) = u_{\gcd(m,n)} \quad \text{for all } m, n \ge 1.$$

The term regularly divisible by all primes is defined in [16] and is shown to be equivalent to that of strongly divisible. We use the following definition which defines the divisibility of the sequence for a set of primes rather than for all primes.

Definition: Let $\{A_n\}_{j=1}^{\infty}$ be a sequence of positive integers. We say that $\{A_n\}_{j=1}^{\infty}$ is regularly divisible with respect to a set of primes $\mathcal{S} = \{p_1, p_2, \ldots\}$, provided that, for each $p \in \mathcal{S}$, $p^i | A_j$ if and only if $r(p^i) | j$, for all $i \ge 1$ and $j \ge 1$, where $r(p^i)$ is the rank of apparition of p^i, that is, $A_{r(p^i)}$ is the first term in the sequence divisible by p^i.

A sequence is said to be regularly divisible if it is regularly divisible by all primes. Since the Fibonacci sequence satisfies the requirements for strong divisibility [9], it is a regularly divisible sequence.

We will use $r = r(2) = 3$ for the rank of apparition of 2. That is, F_r is the first term in the sequence that is divisible by 2. For the rank of apparition of 2^i, we will use $r(2^i) = r_i$..

We will use a generalized base for the positive integers. Since the Fibonacci sequence is regularly divisible by 2, we have that $\frac{r_{i+1}}{r_i}$ is always an integer. Thus, a generalized base $\mathcal{P} = \{1, r, r_2, \ldots, r_i, \ldots\}$ can be used [21] and the number n can be uniquely expressed as

$$n = (n_t n_{t-1} \cdots n_1 n_0)_{\mathcal{P}} = n_t r_t + n_{t-1} r_{t-1} + \cdots + n_1 r + n_0, \text{ where } 0 \le n_i < \frac{r_{i+1}}{r_i}.$$

The version of Kummer's Theorem we need is that in [27]:

Kummer's Theorem for Generalized Binomial Coefficients: Let $\mathcal{A} = \{\mathcal{A}_j\}_{j=1}^{\infty}$ be a sequence of positive integers. If $\mathcal{A}$ is regularly divisible by p, then the highest power of p that divides $\left[{m+n \atop m}\right]_A$ is the number of carries that occur when the integers n and m are added in base $\mathcal{P}$, where $\mathcal{P} = \{r_j\}_{j=0}^{\infty}$ with $r_0 = 1$ and $r_i = r(p^i)$, for all $i \ge 1$.

2. THE FIBONACCI TRIANGLE MODULO 2

One of the interesting results for Pascal's Triangle modulo 2 is that the number of coefficients in the n^{th} row which are congruent to 1 modulo 2, denoted $N[n,2,1]$, is equal to 2^t, where t is the number of ones in n's base two representation [24]. A similar result follows for the Fibonacci triangle.

Theorem 1: For the Fibonacci triangle modulo 2, the number of coefficients in the n^{th} row congruent to 1 modulo 2 is given by $N[n,2,1]=2^t3^s$, where t = number of 1's and s = number of 2's in n's base $\mathcal{P}$ representation.

Proof: The generalized base for the Fibonacci sequence is $\mathcal{P}=\{1,3,6,6,12,\ldots\}$. Since

$$r_{i+1}=\begin{cases} r_i \\ \text{or} \\ 2r_i, \end{cases}$$

for $n=(\ldots n_2n_1n_0)_{\mathcal{P}}$ and $k=(\ldots k_2k_1k_0)_{\mathcal{P}}$, we have that $0\le n_i, k_i<2$ for $i\ge 1$ and $0\le n_0,\ k_0<3$. From Webb & Wells [27], $N[n,2,1]=\prod_{i\ge 0}(n_i+1)$. For no borrow to occur in the base $\mathcal{P}$ subtraction of k from n, there are two choices for k_i for each $n_i=1$, and one choice for each $n_i=0$. If $n_0=2$, there are three choices for k_0. Therefore, $N[n,2,1]=2^t3^s$ where t = number of 1's and s = number of 2's in n's base $\mathcal{P}$ representation.

The following theorem, which is similar to Lucas's theorem for binomial coefficients, provides a way to investigate the iterative behavior of the Fibonacci triangle modulo 2.

Theorem 2: The Fibonacci coefficients satisfy

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{F}} \equiv \prod_{i\ge 1}\binom{n_i}{k_i}\begin{bmatrix} n_0 \\ k_0 \end{bmatrix}_{\mathcal{F}} \pmod 2$$

where

$$\begin{bmatrix} n_0 \\ k_0 \end{bmatrix}_{\mathcal{F}}=0 \text{ for } k_0>n_0 \text{ and } \binom{n_i}{k_i}=0 \text{ for } k_i>n_i.$$

Proof: If a borrow occurs in the base $\mathcal{P}$ subtraction of k from n, then $n_i<k_i$ for some i. Thus, either $\left[\begin{smallmatrix} n_0 \\ k_0 \end{smallmatrix}\right]_{\mathcal{F}}=0$ for $k_0>n_0$ or $\binom{n_i}{k_i}=0$ for some $i\ge 1$ and the result holds trivially.

If no borrow occurs, $0\le k_i\le n_i<2$ for $i\ge 1$, so that

$$\binom{n_i}{k_i}\equiv 1 \pmod 2.$$

For $i=0$, $0\le k_0\le n_0<3$, and

$$\begin{bmatrix} n_0 \\ k_0 \end{bmatrix}_{\mathcal{F}}\equiv 1 \pmod 2.$$

Thus,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{F}} \equiv \prod_{i\ge 1}\binom{n_i}{k_i}\begin{bmatrix} n_0 \\ k_0 \end{bmatrix}_{\mathcal{F}} \pmod 2.$$

Corollary 2.1: For $n = 3h + n_0$ and $m = 3k + k_0$,

$$\begin{bmatrix} n \\ m \end{bmatrix}_{\mathcal{F}} \equiv \binom{h}{k} \begin{bmatrix} n_0 \\ m_0 \end{bmatrix}_{\mathcal{F}} \pmod 2.$$

Proof: Let

$$n = n_t r_t + n_{t-1} r_{t-1} + \cdots + n_1 r + n_0 = 3h + n_0$$

and

$$m = n_t r_t + m_{t-1} r_{t-1} + \cdots + m_1 r + m_0 = 3k + m_0.$$

Since

$$r_{i+1} = \begin{cases} r_i \\ \text{or} \\ 2r_i, \end{cases}$$

the coefficients in the ordinary base 2 expansion of h and k will be from the sets

$$\{n_t, n_{t-1}, \ldots, n_1\} \text{ and } \{m_t, m_{t-1}, \ldots, m_1\}.$$

When $r_i = r_{i+1}$ for some i, such as $r_2 = r_3 = 6$, the base $\mathcal{P}$ requires $n_i, m_i = 0$. The base 2 coefficients of h and k will be n_i and m_i where $r_i \neq r_{i+1}$. Although the exact power of 2 associated with each coefficient can be determined only by looking at the relationship between all the elements in the base, h and k will still have an appropriate base 2 expansion. The residue of $\binom{h}{k}$ modulo 2 will be

$$\binom{h}{k} \equiv \prod_{i \geq 1} \binom{n_i}{m_i} \pmod 2.$$

The above corollary can be used to investigate the iterative behavior of the Fibonacci triangle modulo 2. To begin, we will use the notation of Long [20].

Theorem 3: Let $\Delta_{n,k}$ denote the following triangle,

$$\begin{matrix} & & \begin{bmatrix} 3n \\ 3k \end{bmatrix}_{\mathcal{F}} & & \\ & \begin{bmatrix} 3n+1 \\ 3k \end{bmatrix}_{\mathcal{F}} & & \begin{bmatrix} 3n+1 \\ 3k+1 \end{bmatrix}_{\mathcal{F}} & \\ \begin{bmatrix} 3n+2 \\ 3k \end{bmatrix}_{\mathcal{F}} & & \begin{bmatrix} 3n+2 \\ 3k+1 \end{bmatrix}_{\mathcal{F}} & & \begin{bmatrix} 3n+2 \\ 3k+2 \end{bmatrix}_{\mathcal{F}} \end{matrix}$$

a. The entries in $\Delta_{n,k}$ will be either all congruent to 1 or all congruent to 0 modulo 2. The entries in the Fibonacci triangle not included in one of the triangles $\Delta_{n,k}$ are congruent to 0.

b. The triangles satisfy an element-wise addition modulo 2, $\Delta_{n-1,k-1} + \Delta_{n-1,k} \equiv \Delta_{n,k} \pmod 2$.

c. The Fibonacci triangle of $\Delta_{n,k}$'s is in 1-1 correspondence with Pascal's Triangle modulo 2.

Proof: Since $\left[{t \atop s}\right]_{\mathcal{F}} \equiv 1 \pmod 2$ for $0 \le s \le t < 3$, we have

$$\begin{array}{ccccc} & & \begin{bmatrix}3n\\3k\end{bmatrix}_{\mathcal{F}} & & \\ & \begin{bmatrix}3n+1\\3k\end{bmatrix}_{\mathcal{F}} & & \begin{bmatrix}3n+1\\3k+1\end{bmatrix}_{\mathcal{F}} & \\ \begin{bmatrix}3n+2\\3k\end{bmatrix}_{\mathcal{F}} & & \begin{bmatrix}3n+2\\3k+1\end{bmatrix}_{\mathcal{F}} & & \begin{bmatrix}3n+2\\3k+2\end{bmatrix}_{\mathcal{F}} \end{array} \equiv \begin{array}{ccccc} & & \binom{n}{k}\begin{bmatrix}0\\0\end{bmatrix}_{\mathcal{F}} & & \\ & \binom{n}{k}\begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{F}} & & \binom{n}{k}\begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{F}} & \\ \binom{n}{k}\begin{bmatrix}2\\0\end{bmatrix}_{\mathcal{F}} & & \binom{n}{k}\begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{F}} & & \binom{n}{k}\begin{bmatrix}2\\2\end{bmatrix}_{\mathcal{F}} \end{array}$$

$$\equiv \begin{array}{ccccc} & & \binom{n}{k} & & \\ & \binom{n}{k} & & \binom{n}{k} & \\ \binom{n}{k} & & \binom{n}{k} & & \binom{n}{k} \end{array} \pmod 2.$$

Therefore,

$$\begin{array}{ccccc} & & \begin{bmatrix}3n\\3k\end{bmatrix}_{\mathcal{F}} & & \\ & \begin{bmatrix}3n+1\\3k\end{bmatrix}_{\mathcal{F}} & & \begin{bmatrix}3n+1\\3k+1\end{bmatrix}_{\mathcal{F}} & \\ \begin{bmatrix}3n+2\\3k\end{bmatrix}_{\mathcal{F}} & & \begin{bmatrix}3n+2\\3k+1\end{bmatrix}_{\mathcal{F}} & & \begin{bmatrix}3n+2\\3k+2\end{bmatrix}_{\mathcal{F}} \end{array} \equiv \begin{cases} T_1 = \begin{array}{c}1\\11\\111\end{array} & \text{if } \binom{n}{k} \equiv 1 \\ T_0 = \begin{array}{c}0\\00\\000\end{array} & \text{if } \binom{n}{k} \equiv 0 \end{cases} \pmod 2.$$

The entries not included in one of these triangles are of the form $\left[{3n+t \atop 3k+s}\right]$ with $0 \le t < s \le 2$, and so are congruent to 0 modulo 2.

From Corollary 2.1, we have that

$$\begin{bmatrix}3(n-1)+t\\3(k-1)+s\end{bmatrix}_{\mathcal{F}} + \begin{bmatrix}3(n-1)+t\\3k+s\end{bmatrix}_{\mathcal{F}} \equiv \binom{n-1}{k-1}\begin{bmatrix}t\\s\end{bmatrix}_{\mathcal{F}} + \binom{n-1}{k}\begin{bmatrix}t\\s\end{bmatrix}_{\mathcal{F}} \equiv \binom{n}{k}\begin{bmatrix}t\\s\end{bmatrix}_{\mathcal{F}} \equiv \begin{bmatrix}3n+t\\3k+s\end{bmatrix}_{\mathcal{F}}.$$

Thus, there is an element-wise addition of triangles that satisfies

$$\Delta_{n-1,\,k-1} + \Delta_{n-1,\,k} \equiv \Delta_{n,\,k} \pmod 2.$$

If the identification $T_1 \leftrightarrow 1$ and $T_0 \leftrightarrow 0$ is made, the Fibonacci triangle of Δ's is in 1-1 correspondence with Pascal's Triangle modulo 2.

$$\begin{array}{c} 1\\ 1\ 1\\ 1\ 1\ 1\\ 1\ 0\ 0\ 1\\ 1\ 1\ 0\ 1\ 1\\ 1\ 1\ 1\ 1\ 1\ 1\\ 1\ 0\ 0\ 0\ 0\ 0\ 1\\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \end{array} \quad \leftrightarrow \quad \begin{array}{c} 1\\ \\ \\ 1\quad 1\\ \\ 1\quad 0\quad 1\\ \\ 1\quad 1\quad 1\quad 1 \end{array}$$

Fibonacci Triangle Modulo 2 **Pascal's Triangle Modulo 2**

With this theorem, once the identification with Pascal's Triangle is made, one can see that the pattern continues at all levels of $r(2^t)$. For example, at the level of $r(2^2)=6$, with

$$\begin{matrix} & T_1 & \\ T_1 & & T_1 \end{matrix} \leftrightarrow 1 \quad \text{and} \quad \begin{matrix} & T_0 & \\ T_0 & & T_0 \end{matrix} \leftrightarrow 0,$$

we have the identification shown below

```
                       1
                      1 1
                     1 1 1
                    1 0 0 1
                   1 1 0 1 1
                  1 1 1 1 1 1
                 1 0 0 0 0 0 1
                1 1 0 0 0 0 1 1
               1 1 1 0 0 0 1 1 1
              1 0 0 1 0 0 1 0 0 1
             1 1 0 1 1 0 1 1 0 1 1
            1 1 1 1 1 1 1 1 1 1 1 1
           1 0 0 0 0 0 0 0 0 0 0 0 1
          1 1 0 0 0 0 0 0 0 0 0 0 1 1
         1 1 1 0 0 0 0 0 0 0 0 0 1 1 1
        1 0 0 1 0 0 0 0 0 0 0 0 1 0 0 1
       1 1 0 1 1 0 0 0 0 0 0 0 1 1 0 1 1
      1 1 1 1 1 1 0 0 0 0 0 0 1 1 1 1 1 1
     1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1
    1 1 0 0 0 0 1 1 0 0 0 0 1 1 0 0 0 0 1 1
   1 1 1 0 0 0 1 1 1 0 0 0 1 1 1 0 0 0 1 1 1
  1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1
 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```

Fibonacci Triangle Modulo 2

$\leftrightarrow$

```
   1

  1 1

 1 0 1

1 1 1 1
```

Pascal's Triangle Modulo 2

3. THE LUCAS TRIANGLE MODULO 2

Although the Lucas sequence is not regularly divisible, the structure of the triangle modulo 2 is still determined by the highest power of 2 that divides $[n]!$ defined below. To determine the residues of the coefficients modulo 2, the following lemma will be needed. We will use the notation $2^t \| m$ to mean $2^t | m$ but that $2^{t+1} \nmid m$.

Lemma 1: If $[n]! = L_n L_{n-1} \dots L_2 L_1$, then

$$2^{3k} \| [n]! \text{ for } 3(2k) \le n < 3(2k+1) \quad \text{and} \quad 2^{3k-1} \| [n]! \text{ for } 3(2k-1) \le n < 3(2k).$$

Proof: For the Lucas sequence

$$L_{12} = 322 \equiv 2 \text{ and } L_{13} = 521 \equiv 1 \pmod 8.$$

Thus, the length of the period modulo 8 is 12, because $L_0 = 2$ and $L_1 = 1$. Also since

$$L_n \not\equiv 0 \text{ for } 1 \le n \le 12 \pmod 8,$$

we have that $8 \nmid L_n$ for any n.

Also, as above,

$$L_6 = 18 \equiv 2 \text{ and } L_7 = 29 \equiv 1 \pmod 4,$$

so the length of the period modulo 4 is 6. For $1 \le n \le 6$, $L_n \equiv 0 \pmod 4$ only for $n = 3$. Thus, $L_n \equiv 0 \pmod 4$ for $n = 3 + 6k = 3(2k+1)$, $k \ge 0$. For $1 \le n \le 6$,, $L_n \equiv 2 \pmod 4$ only for $n = 6$. So $2 | L_n$ and $4 \nmid L_n$ for $n = 6k$.

In $[n]!$ there are $\lfloor \frac{n}{3} \rfloor$ factors that are divisible by 2 and $\lfloor \frac{n}{6} \rfloor$ factors that are exactly divisible by 2. Thus, there are $\lfloor \frac{n}{3} \rfloor - \lfloor \frac{n}{6} \rfloor$ that are exactly divisible by 2^2, and so

$$2^{3k} \| [n]! \text{ for } 3(2k-1) \le n < 3(2k+1) \quad \text{and} \quad 2^{3k-1} \| [n]! \text{ for } 3(2k-1) \le n < 3(2k).$$

Theorem 4: The Lucas coefficients satisfy the following congruences.

For $0 \le m \le n$ and $0 \le s \le t \le 2$,

$$\begin{bmatrix} 3n+t \\ 3m+s \end{bmatrix}_{\mathcal{L}} \equiv \begin{cases} \alpha & \text{for } n \text{ even and } m \text{ odd} \\ 1 & \text{otherwise} \end{cases} \pmod 2.$$

For $0 \le m \le n$ and $0 \le t < s \le 2$,

$$\begin{bmatrix} 3n+t \\ 3m+s \end{bmatrix}_{\mathcal{L}} \equiv 0 \pmod 2.$$

Proof: Let e be the highest power of 2 that exactly divides $\left[\begin{smallmatrix} 3n+t \\ 3m+s \end{smallmatrix}\right]_{\mathcal{L}}$. Then $e = e_1 - (e_2 + e_3)$, where $2^{e_1} \| [3n+t]!$, $2^{e_2} \| [3m+s]!$ and $2^{e_3} \| [3(n-m)+(t-s)]!$.

By examining the different cases for n and m odd or even, and applying Lemma 1, we obtain the following values for e.

For $0 \le m \le n$ and $0 \le s \le t \le 2$,

$$e = \begin{cases} -1 & \text{if } n \text{ is even and } m \text{ is odd;} \\ 0 & \text{if } n \text{ is even and } m \text{ is even;} \\ 0 & \text{if } n \text{ is odd and } m \text{ is odd;} \\ 0 & \text{if } n \text{ is odd and } m \text{ is even.} \end{cases}$$

For $0 \le m \le n$ and $0 \le t < s \le 2$,

$$e = \begin{cases} 1 & \text{if } n \text{ is even and } m \text{ is odd;} \\ 1 & \text{if } n \text{ is even and } m \text{ is even;} \\ 1 & \text{if } n \text{ is odd and } m \text{ is odd;} \\ 2 & \text{if } n \text{ is odd and } m \text{ is even.} \end{cases}$$

This theorem can be used to count the number of each of the residues modulo 2 in the n^{th} row of the Lucas triangle and to investigate the iterative patterns in the triangle. The Lucas sequence has the same recurrence relation as the Fibonacci sequence and, like the Fibonacci sequence, satisfies $r(2) = 3$, which is also equal to the period of 2. In determining the number of each of the residues in the n^{th} row of the Lucas triangle, we will use the generalized base corresponding to 2 for the Fibonacci sequence, $\mathcal{P} = \{1, 3, 6, 6, 12, \ldots\}$.

Theorem 5: Let $N[n, 2, a]$ be the number of Lucas coefficients in the n^{th} row congruent to a. For $n = 3h + n_0$, $0 \le n_0 < 3$,

$$N[n,2,1] = \begin{cases} (h+1)(n_0+1) & \text{if } h \text{ is odd,} \\ \left(\frac{h}{2}+1\right)(n_0+1) & \text{if } h \text{ is even,} \end{cases} \qquad N[n,2,\alpha] = \begin{cases} 0 & \text{if } h \text{ is odd,} \\ \left(\frac{h}{2}\right)(n_0+1) & \text{if } h \text{ is even,} \end{cases}$$

and $\quad N[n, 2, 0] = h(2 - n_0)$

Proof: For $n = 3h + n_0$ and $m = 3k + m_0$, if h is odd, then

$$\begin{bmatrix} 3h+n_0 \\ 3k+m_0 \end{bmatrix}_{\mathcal{L}} \equiv 1 \pmod 2,$$

provided $0 \le m_0 \le n_0$. Therefore, there are $h + 1$ choices for k, and there are $n_0 + 1$ choices for m_0. Thus, $N[n, 2, 1] = (h+1)(n_0+1)$.

If h is even, then

$$\begin{bmatrix} 3h+n_0 \\ 3k+m_0 \end{bmatrix}_{\mathcal{L}} \equiv \begin{cases} 1 & \text{for } k \text{ even,} \\ \alpha & \text{for } k \text{ odd.} \end{cases}$$

Thus, there are $\left(\frac{h}{2}+1\right)$ choices for k to be even and $\left(\frac{h}{2}\right)$ choices for k to be odd. There are still (n_0+1) choices for m_0, so that

$$N[n, 2, 1] = \left(\tfrac{h}{2}+1\right)(n_0+1) \text{ and } N[n, 2, \alpha] = \left(\tfrac{h}{2}\right)(n_0+1), \text{ for } h \text{ even.}$$

If $0 \le n_0 < m_0 \le 2$, then

$$\begin{bmatrix} 3h+n_0 \\ 3k+m_0 \end{bmatrix}_{\mathcal{L}} \equiv 0 \pmod 2.$$

There are h choices for k and $(2-n_0)$ choices for m_0, so that $N[n, 2, 0] = h(2-n_0)$.

Theorem 6: For $0 \le m \le n$, the entries in the Lucas triangle denoted $\Delta_{n,m}$,

$$\begin{matrix} & & \begin{bmatrix} 3n \\ 3m \end{bmatrix}_{\mathcal{L}} & & \\ & \begin{bmatrix} 3n+1 \\ 3m \end{bmatrix}_{\mathcal{L}} & & \begin{bmatrix} 3n+1 \\ 3m+1 \end{bmatrix}_{\mathcal{L}} & \\ \begin{bmatrix} 3n+2 \\ 3m \end{bmatrix}_{\mathcal{L}} & & \begin{bmatrix} 3n+2 \\ 3m+1 \end{bmatrix}_{\mathcal{L}} & & \begin{bmatrix} 3n+2 \\ 3m+2 \end{bmatrix}_{\mathcal{L}} \end{matrix}$$

are either all congruent to one or all congruent to α modulo 2. The entries not included in these triangles are congruent to zero modulo 2.

Proof: From Theorem 4, it follows directly that the entries in the initial triangles are all congruent to α modulo 2 if n is even and m is odd. Otherwise, all entries are congruent to 1. The entries not included in these triangles are $\left[\begin{smallmatrix} 3n+t \\ 3m+s \end{smallmatrix}\right]_{\mathcal{L}}$, where $0 \le t < s \le 2$, and so are congruent to zero modulo 2.

Theorem 7: For $r_i = 2^{i-1}3$, let $\Delta_{n,m}$ denote the following entries in the Lucas triangle,

$$\begin{matrix} & \begin{bmatrix} nr_i \\ mr_i \end{bmatrix}_{\mathcal{L}} & \\ \begin{bmatrix} nr_i+1 \\ mr_i \end{bmatrix}_{\mathcal{L}} & & \begin{bmatrix} nr_i+1 \\ mr_i+1 \end{bmatrix}_{\mathcal{L}} \\ \cdot^{\cdot^{\cdot}} & & \ddots \\ \begin{bmatrix} nr_i+r_i-1 \\ mr_i \end{bmatrix}_{\mathcal{L}} & & \begin{bmatrix} nr_i+r_i-1 \\ mr_i+r_i-1 \end{bmatrix}_{\mathcal{L}}, \end{matrix}$$

and let $\nabla_{n,m}$ denote the entries not included in one of these triangles.

a. For $i = 1$, the initial triangles, $\Delta_{n,m}$, $\Delta_{n,m+1}$, $\Delta_{n+1,m+1}$, do not satisfy an element-wise addition modulo 2 as in the Fibonacci triangle.

b. For $i > 1$, the triangles satisfy

$$\Delta_{n,m} \equiv \Delta_{n,m+1} \equiv \Delta_{n+1,m+1} \equiv \Delta_{0,0}$$
$$\nabla_{n,m} \equiv \nabla_{n,m+1} \equiv \nabla_{n+1,m+1}.$$

Proof: For $i = 1$, from the Lucas triangle modulo 2, we can see that

$$\Delta_{1,0} + \Delta_{1,1} \not\equiv \Delta_{2,1} \pmod 2$$
$$\Delta_{5,2} + \Delta_{5,3} \not\equiv \Delta_{6,3} \pmod 2.$$

Thus, the initial triangles do not satisfy an element-wise addition modulo 2.

For $i > 1$ and $0 \le h, k \le 2^{i-1} - 1$, h and k determine whether $2^{i-1}n + h$ and $2^{i-1}m + k$ are odd or even, so that

$$\begin{bmatrix} nr_i + 3h + t \\ mr_i + 3k + s \end{bmatrix}_{\mathcal{L}} = \begin{bmatrix} 3(2^{i-1}n + h) + t \\ 3(2^{i-1}m + k) + s \end{bmatrix}_{\mathcal{L}} \equiv \begin{bmatrix} 3h + t \\ 3k + s \end{bmatrix}_{\mathcal{L}} \pmod 2.$$

Thus,

$$\begin{bmatrix} nr_i + 3h + t \\ mr_i + 3k + s \end{bmatrix}_{\mathcal{L}} \equiv \begin{bmatrix} nr_i + 3h + t \\ (m+1)r_i + 3k + s \end{bmatrix}_{\mathcal{L}} \equiv \begin{bmatrix} (n+1)r_i + 3h + t \\ (m+1)r_i + 3k + s \end{bmatrix}_{\mathcal{L}} \equiv \begin{bmatrix} 3h + t \\ 3k + s \end{bmatrix}_{\mathcal{L}} \pmod 2.$$

Therefore,

$$\Delta_{n,m} \equiv \Delta_{n,m+1} \equiv \Delta_{n+1,m+1} \equiv \Delta_{0,0} \quad \text{and} \quad \nabla_{n,m} \equiv \nabla_{n,m+1} \equiv \nabla_{n+1,m+1}.$$

From Theorem 7, the Lucas triangle of Δs with $i = 1$ has initial triangles

$$T_1 = \begin{matrix} 1 \\ 11 \\ 111 \end{matrix} \quad \text{and} \quad T_\alpha = \begin{matrix} \alpha \\ \alpha\alpha \\ \alpha\alpha\alpha \end{matrix}.$$

Using the identification $T_1 \leftrightarrow 1$ and $T_\alpha \leftrightarrow \alpha$, the pattern in the Lucas triangle becomes more apparent.

$$\begin{array}{ccc}
1 & & 1 \\
1\;1 & & \\
1\;1\;1 & & \\
1\;0\;0\;1 & & \\
1\;1\;0\;1\;1 & & 1\quad 1 \\
1\;1\;1\;1\;1\;1 & & \\
1\;0\;0\;\alpha\;0\;0\;1 & & \\
1\;1\;0\;\alpha\;\alpha\;0\;1\;1 & & 1\quad \alpha\quad 1 \\
1\;1\;1\;\alpha\;\alpha\;\alpha\;1\;1\;1 & & \\
1\;0\;0\;1\;0\;0\;1\;0\;0\;1 & \leftrightarrow & \\
1\;1\;0\;1\;1\;0\;1\;1\;0\;1\;1 & & 1\quad 1\quad 1\quad 1 \\
1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1 & & \\
1\;0\;0\;\alpha\;0\;0\;1\;0\;0\;\alpha\;0\;0\;1 & & \\
1\;1\;0\;\alpha\;\alpha\;0\;1\;1\;0\;\alpha\;\alpha\;0\;1\;1 & & 1\quad \alpha\quad 1\quad \alpha\quad 1 \\
1\;1\;1\;\alpha\;\alpha\;\alpha\;1\;1\;1\;\alpha\;\alpha\;\alpha\;1\;1\;1 & & \\
1\;0\;0\;1\;0\;0\;1\;0\;0\;1\;0\;0\;1\;0\;0\;1 & & \\
1\;1\;0\;1\;1\;0\;1\;1\;0\;1\;1\;0\;1\;1\;0\;1\;1 & & 1\quad 1\quad 1\quad 1\quad 1\quad 1 \\
1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1 & &
\end{array}$$

Lucas Triangle Modulo 2

Also from Theorem 7, we see that this pattern does not continue for $i > 1$. For example, with $i = 2$, if the following correspondence is made,

$$T_1 = \begin{array}{c} 1 \\ 11 \\ 111 \\ 1001 \\ 11011 \\ 111111 \end{array} \leftrightarrow 1,$$

then the Lucas triangle modulo 2 can be associated with a triangle of all ones. That is, the initial triangle will be the only triangle repeated.

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 1\ 1 \\ 1\ 0\ 0\ 1 \\ 1\ 1\ 0\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1 \\ 1\ 0\ 0\ \alpha\ 0\ 0\ 1 \\ 1\ 1\ 0\ \alpha\ \alpha\ 0\ 1\ 1 \\ 1\ 1\ 1\ \alpha\ \alpha\ \alpha\ 1\ 1\ 1 \\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1 \\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\ 1\ 0\ 0\ \alpha\ 0\ 0\ 1\ 0\ 0\ \alpha\ 0\ 0\ 1 \\ 1\ 1\ 0\ \alpha\ \alpha\ 0\ 1\ 1\ 0\ \alpha\ \alpha\ 0\ 1\ 1 \\ 1\ 1\ 1\ \alpha\ \alpha\ \alpha\ 1\ 1\ 1\ \alpha\ \alpha\ \alpha\ 1\ 1\ 1 \\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1 \\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \end{array} \quad \leftrightarrow \quad \begin{array}{c} 1 \\ \\ 1\ \ 1 \\ \\ 1\ \ 1\ \ 1 \end{array}$$

Lucas Triangle Modulo 2

4. THE RELATIONSHIP BETWEEN THE FIBONACCI AND LUCAS TRIANGLES MODULO 2

We can use Theorem 2 and Theorem 4 to look at the relationship between the Fibonacci triangle and the Lucas triangle modulo 2.

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 1\ 1 \\ 1\ 0\ 0\ 1 \\ 1\ 1\ 0\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1 \\ 1\ 0\ 0\ \alpha\ 0\ 0\ 1 \\ 1\ 1\ 0\ \alpha\ \alpha\ 0\ 1\ 1 \\ 1\ 1\ 1\ \alpha\ \alpha\ \alpha\ 1\ 1\ 1 \\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1 \\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\ 1\ 0\ 0\ \alpha\ 0\ 0\ 1\ 0\ 0\ \alpha\ 0\ 0\ 1 \\ 1\ 1\ 0\ \alpha\ \alpha\ 0\ 1\ 1\ 0\ \alpha\ \alpha\ 0\ 1\ 1 \\ 1\ 1\ 1\ \alpha\ \alpha\ \alpha\ 1\ 1\ 1\ \alpha\ \alpha\ \alpha\ 1\ 1\ 1 \\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1 \\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \end{array}$$

Lucas Triangle Modulo 2

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 1\ 1 \\ 1\ 0\ 0\ 1 \\ 1\ 1\ 0\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1 \\ 1\ 0\ 0\ 0\ 0\ 0\ 1 \\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1 \\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1 \\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1 \\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1 \\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1 \\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1 \\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1 \end{array}$$

Fibonacci Triangle Modulo 2

Theorem 8: The Fibonacci and Lucas coefficients satisfy the following relationships modulo 2:

If $\begin{bmatrix} n \\ m \end{bmatrix}_{\mathscr{F}} \equiv 1$, then $\begin{bmatrix} n \\ m \end{bmatrix}_{\mathscr{L}} \equiv 1$.

If $\begin{bmatrix} n \\ m \end{bmatrix}_{\mathscr{F}} \equiv 0$, then $\begin{bmatrix} n \\ m \end{bmatrix}_{\mathscr{L}} \equiv \begin{cases} 0 & \text{if a borrow occurs in the } n_0 \text{ position,} \\ \alpha & \text{if a borrow occurs in the } n_1 \text{ position,} \\ 1 & \text{all other borrows.} \end{cases}$

If $\begin{bmatrix} n \\ m \end{bmatrix}_{\mathscr{L}} \equiv 0$, then $\begin{bmatrix} n \\ m \end{bmatrix}_{\mathscr{F}} \equiv 0$.

If $\begin{bmatrix} n \\ m \end{bmatrix}_{\mathcal{L}} \equiv \alpha$, then $\begin{bmatrix} n \\ m \end{bmatrix}_{\mathcal{F}} \equiv 0$.

If $\begin{bmatrix} n \\ m \end{bmatrix}_{\mathcal{L}} \equiv 1$, then $\begin{bmatrix} n \\ m \end{bmatrix}_{\mathcal{F}} \equiv \begin{cases} 0 & \text{if a borrow occurs,} \\ 1 & \text{if no borrow occurs.} \end{cases}$

Proof: For $n = 3h + n_0 = (\dots n_2 n_1 n_0)_{\mathcal{P}}$ and $m = 3k + m_0 = (\dots m_2 m_1 m_0)_{\mathcal{P}}$, if $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_{\mathcal{F}} \equiv 1 \pmod 2$, then $m_1 \le n_1 < 2$ and $m_0 \le n_0 < 3$. Thus, if $n_1 = 1$, h is odd and $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_{\mathcal{L}} \equiv 1 \pmod 2$.

If $n_1 = 0$, then $k_1 = 0$ and h and k are even, so that $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_{\mathcal{L}} \equiv 1 \pmod 2$.

If $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_{\mathcal{F}} \equiv 0 \pmod 2$, then a borrow occurs. If the borrow occurs in the n_0 position, $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_{\mathcal{L}} \equiv 0 \pmod 2$. If the borrow occurs in the n_1 position, then h is even and k is odd. Thus, $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_{\mathcal{L}} \equiv \alpha \pmod 2$. For all other borrows, $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_{\mathcal{L}} \equiv 1 \pmod 2$.

If $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_{\mathcal{L}} \equiv 0 \pmod 2$, then $0 \le n_0 < m_0 < 3$. Thus, a borrow occurs in the base $\mathcal{P}$ subtraction of m from n. Therefore, $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_{\mathcal{F}} \equiv 0 \pmod 2$.

If $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_{\mathcal{L}} \equiv \alpha \pmod 2$ implies h is even and k is odd, which occurs only if $n_1 = 0$ and $m_1 = 1$. This means a borrow will occur in the base $\mathcal{P}$ subtraction of m from n and $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_{\mathcal{F}} \equiv 0 \pmod 2$.

If $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_{\mathcal{L}} \equiv 1 \pmod 2$, then no borrow occurs in the n_0 or n_1 positions. However, a borrow may occur in other positions. Thus,

$$\begin{bmatrix} n \\ m \end{bmatrix}_{\mathcal{F}} \equiv \begin{cases} 0 & \text{if a borrow occurs,} \\ 1 & \text{if no borrow occurs.} \end{cases}$$

5. CONCLUSION

The iterative patterns in the Fibonacci triangle and Pascal's Triangle modulo 2 are similar except for the initial triangles that are repeated in both. For the Fibonacci triangle, the initial triangle is

$$T = \begin{matrix} 1 \\ 11 \\ 111 \end{matrix}$$

and for Pascal's Triangle, the initial triangle is

$$T = \begin{matrix} 1 \\ 11 \end{matrix}.$$

These triangles arise because $r(2)=3$ for the Fibonacci case, which also equals the period modulo 2 for the Fibonacci sequence and $r(2) = 2$ for the Pascal case, which also equals the period modulo 2 for the positive integers. If we look at all second-order sequences, $u_n = au_{n-1} + bu_{n-2}$ with initial conditions $u_0 = 0$ and $u_1 = 1$, they can be categorized into four types.

1. For $a \equiv 0, b \equiv 1 \pmod 2$, $u_n \equiv u_{n-2} \pmod 2$, for $n \ge 2$ and $r(2) = 2$ which equals the period of 2.
2. For $a, b \equiv 1 \pmod 2$, $u_n \equiv u_{n-1} \pmod 2$, for $n \ge 2$ and $r(2) = 3$ which equals the period of 2.

3. For $a \equiv 1, b \equiv 0 \pmod 2$, $u_n \equiv u_{n-1} \pmod 2$, for $n \geq 2$. The prime 2 does not occur as a factor.
4. For $a, b \equiv 0 \pmod 2$, $u_n \equiv 0 \pmod 2$, for $n \geq 2$. All terms are divisible by 2.

This means there are only four distinct triangles modulo 2 formed by the generalized coefficients,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{u_n u_{n-1} \cdots u_1}{(u_k u_{k-1} \cdots u_1)(u_{n-k} u_{n-k-1} \cdots u_1)}.$$

1. Pascal's Triangle comes from type 1 sequences:

```
   1
   11
  101
  1111
 10001
 110011
1010101
11111111
```

2. The Fibonacci triangle comes from type 2 sequences:

```
    1
    11
   111
   1001
  11011
  111111
 1000001
 11000011
111000111
```

3. A triangle of 1's comes from type 3 sequences:

```
   1
   11
  111
  1111
 11111
 111111
1111111
11111111
```

4. A triangle of 0's comes from type 4 sequences:

```
   1
   11
  101
  1001
 10001
 100001
1000001
10000001
```

Thus, Pascal's Triangle and the Fibonacci triangle are the only two significant triangles modulo 2. They only differ by the repetition of the initial triangle. When the initial conditions are changed, the sequence is no longer regularly divisible. The triangles of coefficients from these sequences, such as the Lucas triangle, do not have the same iterative behavior as Pascal's Triangle.

REFERENCES

1. N. J. Fine. "Binomial Coefficients Modulo A Prime." *Amer. Math. Monthly* **14** (1947):598-592.
2. H. W. Gould. "The Bracket Function and Fontené-Ward Generalized Binomial Coefficients with Application to Fibonomial Coefficients." *The Fibonacci Quarterly* **7.1** (1969):23-40.
3. H. W. Gould & W. E. Greig. "A Lucas Triangle Primality Criterion Dual to that of Mann-Shanks." *The Fibonacci Quarterly* **23.1** (1985):66-69.
4. H. Gupta. "On a Problem in Parity." *Indian J. Math.* **11** (1969):157-63.

5. H. Harborth. "Number of Odd Binomial Coefficients." *Proc. Amer. Math. Soc.* **62** (1977): 19-22.
6. E. Hexel & H. Sachs. "Counting Residues Modulo A Prime in Pascal's Triangle." *Indian J. Math.* **20** (1978):91-105.
7. B. R. Hodgson. "On Some Number Sequences Related to the Parity of Binomial Coefficients." *The Fibonacci Quarterly* **30.1** (1992):35-47.
8. V. E. Hoggatt, Jr. "Fibonacci Numbers and Generalized Binomial Coefficients." *The Fibonacci Quarterly* **5.5** (1967):383-400.
9. P. Horak & L. Skula. "A Characterization of the Second-Order Strong Divisibility Sequences." *The Fibonacci Quarterly* **23.2** (1985):126-32.
10. H. Hosoya. "Fibonacci Triangle." *The Fibonacci Quarterly* **14.2** (1976):173-79.
11. F. T. Howard. "The Number of Binomial Coefficients Divisible by a Fixed Power of 2." *Proc. Amer. Math. Soc.* **28** (1971):236-42.
12. E. Jacobson. "The Distribution of Residues of Two-Term Recurrence Sequences." *The Fibonacci Quarterly* **28.3** (1990):227-29.
13. N. Jensen. "Some Results of Divisibility Sequences." In *Applications of Fibonacci Numbers*, pp. 181-89. Dordrecht: Kluwer, 1990.
14. S. K. Jerbic. "Fibonomial Coefficients—A Few Summation Properties." Master's Thesis, San Jose State College, San Jose, CA 1968.
15. C. Kimberling. "Strong Divisibility Sequences and Some Conjectures." *The Fibonacci Quarterly* **17.1** (1979):13-17.
16. D. E. Knuth & H. S. Wilf. "The Power of a Prime that Divides a Generalized Binomial Coefficient." *J. Reine Angew. Math.* **396** (1989):212-19.
17. S. H. L. Kung. "Parity Triangles of Pascal's Triangle." *The Fibonacci Quarterly* **14.1** (1976):54.
18. D. A. Lind. "A Determinant Involving Generalized Binomial Coefficients." *The Fibonacci Quarterly* **9.2** (1971):113-19, 162.
19. C. T. Long. "Some Divisibility Properties of Pascal's Triangle." *The Fibonacci Quarterly* **19.3** (1981):257-63.
20. C. T. Long. "Pascal's Triangle Modulo p." *The Fibonacci Quarterly* **19.5** (1981):458-63.
21. C. T. Long & N. Woo. "On Bases for the Set of Integers." *Duke Math. J.* **38** (1971):583-590.
22. M. D. McIlroy. "The Number of 1's in Binary Integers: Bounds and Extremal Properties." *SIAM J. Comput.* **3** (1974):225-61.
23. D. J. Newman. "On the Number of Binary Digits in a Multiple of Three." *Proc. Amer. Math. Soc.* **21** (1969):719-21.
24. D. Singmaster. "Divisibility of Binomial and Multinomial Coefficients by Primes and Prime Powers." In *A Collection of Manuscripts Related to Fibonacci Sequences*, pp. 98-113. Santa Clara, CA: The Fibonacci Association, 1980.
25. K. B. Stolarsky. "Powers and Exponential Sums of Digital Sums Related to Binomial Coefficient Parity." *SIAM J. Appl. Math.* **32** (1977):717-30.
26. W. A. Webb. "The Number of Binomial Coefficients in Residue Classes Modulo p and p^2." *Colloquium Mathematicum* **60/61** (1990):275-80.
27. W. A. Webb & D. Wells. "Kummer's Theorem for Generalized Binomial Coefficients." Submitted for publication.

AMS Classification Numbers: 11B39, 11B65, 11A07