POWERS OF DIGITAL SUMS

Tom C. Brown
Simon Fraser University, Burnaby, B.C., Canada
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1. INTRODUCTION

Let $s(n)$ denote the sum of the base 10 digits of the nonnegative integer n, and let $\log x$ denote the base 10 logarithm of x. R. E. Kennedy and C. Cooper have shown [1] that for any positive integer k,

$$\frac{1}{x}\sum_{n\le k} s(n)^k = \left(\frac{9}{2}\right)^k \log^k x + O(\log^{k-\frac{1}{3}} x),$$

and they conjectured that for any positive integer k,

$$\frac{1}{x}\sum_{n\le k} s(n)^k = \left(\frac{9}{2}\right)^k \log^k x + O(\log^{k-1} x).$$

Recently [2] the same authors have shown, providing some evidence for the truth of this conjecture, that for each fixed positive integer k,

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^k = \left(\frac{9}{2}\right)^k n^k + O(n^{k-1}).$$

In this note, we extend the result just mentioned. When k is a fixed positive integer, we show that for each m it is true that

$$\frac{1}{x}\sum_{n\le k} s(n)^k = \left(\frac{9}{2}\right)^k \log^k x + O(\log^{k-1} x),$$

provided that x is restricted to the set of those positive integers having at most m nonzero digits in their base 10 representations. (Thus, the Kennedy & Cooper result is exactly the case $m = 1$.) We use the Kennedy & Cooper result in the course of our proof.

We state our result in the following form.

Proposition: Let $m \ge 1$ and $k \ge 1$ be fixed integers. Then there is a constant $A = A(k, m)$ such that if x is a positive integer with at most m nonzero digits in its base 10 representation,

$$\left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k (\log x)^k\right| < A(\log x)^{k-1}.$$

2. REMARKS AND LEMMAS

Remark 1: It is easy to check that if m, k are fixed positive integers and

$$\left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k [\log x]^k\right| < c[\log x]^{k-1},$$

then

$$\left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k (\log x)^k\right| < d(\log x)^{k-1},$$

where [ ] denotes the greatest integer function and d is a constant that depends only on c and k.

To see this, suppose $10^n \le x \le 10^{n+1}$, so that $n = [\log x]$. Let $\log x = n + \alpha$, where $0 \le \alpha < 1$. Then

$$\left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k (n+\alpha)^k\right| < \left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k n^k\right| + \left|\left(\frac{9}{2}\right)^k n^k - \left(\frac{9}{2}\right)^k (n+\alpha)^k\right|$$

$$= \left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k (n+\alpha)^k\right| + \left(\frac{9}{2}\right)^k \left\{k\alpha n^{k-1} + \binom{k}{2}\alpha^2 n^{k-2} + \cdots + \binom{k}{k}\alpha^k\right\}$$

$$< cn^{k-1} + c'n^{k-1} = dn^{k-1} \le d(n+\alpha)^{k-1}.$$

Remark 2: In view of Remark 1, to prove the Proposition above, it is sufficient (and convenient) to prove the following statement, which will be done by induction on m.

For fixed positive integers m, k, there is an $A = A(k, m)$ such that if $10^n \le x < 10^{n+1}$ and x has at most m nonzero digits in its base 10 representation, then

$$\left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k n^k\right| < An^{k-1}.$$

In the following three lemmas, k, n, p, y, and t all denote integers.

Lemma 1: For each $k \ge 1$, there is an $A(k)$ such that

$$\frac{n\cdot 10^{p+1}}{10^n + 10^{p+1}}\left(1 - \left(\frac{p}{n}\right)^k\right) < A(k) \quad \text{for all } n, p \text{ with } 1 \le p \le n.$$

Proof: Let $s = n - p - 1$. Then $-1 \le s \le n-2$ and

$$\frac{n\cdot 10^{p+1}}{10^n + 10^{p+1}}\left(1 - \left(\frac{p}{n}\right)^k\right) = \frac{n}{10^s+1}\left(1 - \left(1 - \frac{s+1}{n}\right)^k\right)$$

$$= \frac{n}{10^s+1}\left(k\cdot\frac{s+1}{n} - \binom{k}{2}\frac{(s+1)^2}{n^2} + \binom{k}{3}\frac{(s+1)^3}{n^3} - \cdots - (-1)^k\binom{k}{k}\frac{(s+1)^k}{n^k}\right)$$

$$= \frac{k\cdot(s+1)}{10^s+1} + o(1) < A(k).$$

Note that for $i \ge 2$,

$$\frac{n}{10^s+1}\cdot\frac{(s+1)^i}{n^i} \le \left(\max_{-1\le s<\infty}\frac{(s+1)^i}{10^s+1}\right)\cdot\frac{1}{n^{i-1}} \to 0 \text{ as } n \to \infty.$$

Lemma 2: Fix $k \geq 1$. Let $A(k)$ be as in Lemma 1. Then for any y in the interval $10^p \leq y < 10^{p+1} \leq 10^n$ and any $t \geq 1$,

$$\left| \frac{t \cdot 10^n}{t \cdot 10^n + y}\left(\frac{9}{2}\right)^k n^k + \frac{y}{t \cdot 10^n + y}\left(\frac{9}{2}\right)^k p^k - \left(\frac{9}{2}\right)^k n^k \right| < \left(\frac{9}{2}\right)^k A(k) n^{k-1}.$$

Proof:

$$\left| \frac{t \cdot 10^n}{t \cdot 10^n + y} n^k + \frac{y}{t \cdot 10^n + y} p^k - n^k \right| = n^k \frac{y}{t \cdot 10^n + y}\left(1 - \left(\frac{p}{n}\right)^k\right)$$

$$< n^k \frac{10^{p+1}}{t \cdot 10^n + 10^{p+1}} \cdot \left(1 - \left(\frac{p}{n}\right)^k\right) \leq n^{k-1} \cdot \frac{n \cdot 10^{p+1}}{10^n + 10^{p+1}}\left(1 - \left(\frac{p}{n}\right)^k\right) < A(k) \cdot n^{k-1},$$

where the last inequality is given by Lemma 1.

Lemma 3: Let $t \geq 1$ and $k \geq 1$ be fixed. Then

$$\frac{1}{t \cdot 10^n} \sum_{i=0}^{t \cdot 10^n - 1} s(i)^k = \left(\frac{9}{2}\right)^k n^k + O(n^{k-1}).$$

In other words, there is $B(t, k)$ with

$$\left| \frac{1}{t \cdot 10^n} \sum_{i=0}^{t \cdot 10^n - 1} s(i)^k - \left(\frac{9}{2}\right)^k n^k \right| < B(t, k) n^{k-1}.$$

To prove the Proposition, we will only need this lemma for $t = 1, 2, \ldots, 9$.

Proof: We use induction on t. For $t = 1$, this is the result of Kennedy & Cooper mentioned above. Now fix $t \geq 1$ and assume the result for this t. Then, using the fact that $s(t \cdot 10^n + i) = s(t) + s(i)$, $0 \leq i \leq 10^n - 1$,

$$\left| \frac{1}{(t+1) \cdot 10^n} \sum_{i=0}^{(t+1) \cdot 10^n - 1} s(i)^k - \left(\frac{9}{2}\right)^k \right| \leq \left| \frac{t}{t+1}\left(\frac{1}{t \cdot 10^n} \sum_{i=0}^{t \cdot 10^n - 1} s(i)^k - \left(\frac{9}{2}\right)^k\right) \right|$$

$$+ \left| \frac{1}{t+1}\left(\frac{1}{10^n} \sum_{i=0}^{10^n - 1} s(i)^k - \left(\frac{9}{2}\right)^k\right) \right| + \frac{1}{t+1} \cdot \frac{1}{10^n} \sum_{i=0}^{10^n - 1} \left(c_1 s(i)^{k-1} + c_2 s(i)^{k-2} + \cdots + c_k\right)$$

$$< \frac{t}{t+1} B(t, k) n^{k-1} + \frac{1}{t+1} B(1, k) n^{k-1} + C n^{k-1} < B(t+1, k) n^{k-1}.$$

Note that we used the result of Kennedy & Cooper a second time. Here $c_1, \ldots, c_k$, C are constants that depend only on k and t.

3. PROOF OF THE PROPOSITION

According to Remark 2, we need to show that, for each $m \geq 1$ and $k \geq 1$, there is a constant $A(k, m)$ such that if $10^n \leq x < 10^{n+1}$ and x has at most m nonzero digits in its base 10 representation, then

$$\left| \frac{1}{x} \sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k n^k \right| < A(k, m) n^{k-1}.$$

If $m = 1$, this follows from Lemma 3 (for all k), since then $x = t \cdot 10^n$, $1 \le t \le 9$.

Now assume the result for a given $m \ge 1$, and let x have $m+1$ nonzero digits, say,

$$10^n \le x < 10^{n+1}, \;\; x = t \cdot 10^n + y, \;\; 1 \le t \le 9, \;\; 10^p \le y < 10^{p+1} \le 10^n,$$

where y has m nonzero digits. Then, using $s(t \cdot 10^n + i) = t + s(i)$,

$$\left| \frac{1}{x} \sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k \right| \le \left| \frac{t \cdot 10^n}{t \cdot 10^n + y} \left(\frac{1}{t \cdot 10^n} \sum_{i=0}^{t \cdot 10^n - 1} s(i)^k - \left(\frac{9}{2}\right)^k \right) \right|$$

$$+ \left| \frac{y}{t \cdot 10^n + y} \left(\frac{1}{y} \sum_{i=0}^{y-1} s(i)^k - \left(\frac{9}{2}\right)^k \right) \right| + \frac{y}{t \cdot 10^n + y} \cdot \frac{1}{y} \sum_{i=0}^{y-1} \left(c_1 s(i)^{k-1} + c_2 s(i)^{k-2} + \cdots + c_k \right)$$

$$< A(k, 1) n^{k-1} + A(k, m) p^{k-1} + D p^{k-1} < A(k, m+1) n^{k-1}.$$

Here $c_1, \ldots, c_k$, D are constants that depend on k and t, but since $1 \le t \le 9$, they in fact depend only on k. For the second equality, we used Lemma 3 as well as the induction hypothesis.

REFERENCES

1. C. Cooper & R. E. Kennedy. "Digit Sum Sums." *J. Inst. Math. Comp. Sci.* **5** (1992):45-49.
2. C. Cooper & R. E. Kennedy. "Sums of Powers of Digital Sums." *The Fibonacci Quarterly* **31.4** (1993):341-45.

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