FORMULAS FOR $1+2^p+3^p+\cdots+n^p$

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1. INTRODUCTION

Let $S_p(n)=1+2^p+3^p+\cdots+n^p$, with n and p positive integers. In [3] R. A. Khan, using the binomial theorem and a definite integral, gave a proof of the general recurrence formula for $S_1(n), S_2(n), S_3(n), \ldots$ in terms of powers of n. A matrix formula for $S_p(n)$, obtained by solving a difference equation by matrix methods, is given in [2].

In this note the recurrence formulas in terms of powers of n and of $n+1$ are given in symbolic form, only using the binomial theorem. These formulas are both easily remembered and applied.

These recurrence formulas are then used to establish, employing Cramer's rule, explicit expressions for $S_p(n)$ in determinant form.

Finally, the usual formulas for $S_p(n)$ as polynomials of degree $p+1$ in n and $n+1$, with coefficients in terms of the Bernoulli numbers, are derived from these determinants. It is noted that it is possible to do this without prior knowledge of the Bernoulli numbers.

2. FORMULAS IN TERMS OF POWERS OF n

2.1 A Recurrence Formula

Let $n\in N$, with N the set of positive integers. For $k\in N$, let

$$S_k(n)=1+2^k+3^k+\cdots+n^k=\sum_{r=1}^{n}r^k,$$

$S_k(0)=0$, and take $S_0(n)=n$. Then

$$\begin{aligned}
n^k &= S_k(n)-S_k(n-1)\\
&= S_k(n)-\sum_{r=1}^{n}(r-1)^k\\
&= S_k(n)-\sum_{r=1}^{n}\left(\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}r^i\right)\\
&= S_k(n)-\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}S_i(n) \qquad (2.1.1)\\
&= -\sum_{i=0}^{k-1}\binom{k}{i}(-1)^{k-i}S_i(n). \qquad (2.1.2)
\end{aligned}$$

The equation

$$S^k-(S-1)^k=n^k, \tag{2.1.3}$$

in which the binomial power is expanded and S^i $(i=0,1,2,\ldots,k)$ are then replaced by $S_i(n)$, provides a mnemonic for (2.1.1).

For example, for $k=2$, formula (2.1.3) yields

$$S^2-(S^2-2S+1)=n^2,$$

and thus

$$2S_1(n)-n=n^2,$$

giving the well-known result

$$S_1(n)=\frac{n}{2}(n+1). \tag{2.1.4}$$

Next, for $k=3$, it similarly follows that

$$3S_2(n)-3S_1(n)+n=n^3,$$

so that substitution of (2.1.4) leads to the formula

$$S_2(n)=\frac{n}{6}(n+1)(2n+1).$$

2.2 $S_p(n)$ as a Determinant

Let $p\in N$ and let $k=1,2,\ldots,p,p+1$ in (2.1.2). Then, solving for $S_p(n)$ in the resulting $(p+1)\times(p+1)$ lower triangular linear system by means of Cramer's rule, the determinant representation

$$S_p(n)=\frac{1}{(p+1)!}\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & n \\ -1 & \binom{2}{1} & 0 & 0 & \cdots & 0 & n^2 \\ 1 & -\binom{3}{1} & \binom{3}{2} & 0 & \cdots & 0 & n^3 \\ \vdots & \vdots & \vdots & & & & \vdots \\ (-1)^{p-1} & (-1)^p\binom{p}{1} & (-1)^{p+1}\binom{p}{2} & & \cdots & \binom{p}{p-1} & n^p \\ (-1)^p & (-1)^{p+1}\binom{p+1}{1} & (-1)^{p+2}\binom{p+1}{2} & & \cdots & -\binom{p+1}{p-1} & n^{p+1} \end{vmatrix} \tag{2.2.1}$$

$$=p!\begin{vmatrix} \frac{1}{1!} & 0 & 0 & 0 & \cdots & 0 & \frac{n}{1!} \\ -\frac{1}{2!} & \frac{1}{1!} & 0 & 0 & \cdots & 0 & \frac{n^2}{2!} \\ \frac{1}{3!} & -\frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 & \frac{n^3}{3!} \\ \vdots & \vdots & \vdots & & & & \vdots \\ \frac{(-1)^{p-1}}{p!} & \frac{(-1)^p}{(p-1)!} & \frac{(-1)^{p+1}}{(p-2)!} & & \cdots & \frac{1}{1!} & \frac{n^p}{p!} \\ \frac{(-1)^p}{(p+1)!} & \frac{(-1)^{p+1}}{p!} & \frac{(-1)^{p+2}}{(p-1)!} & & \cdots & -\frac{1}{2!} & \frac{n^{p+1}}{(p+1)!} \end{vmatrix} \tag{2.2.2}$$

is obtained. The step from (2.2.1) to (2.2.2) follows by first multiplying the i^{th} row of the determinant by $1/i!$ for $i=1,2,\ldots,p+1$, and then multiplying the j^{th} column of the resulting determinant by $(j-1)!$ for $j=2,3,\ldots,p$.

2.3 $S_p(n)$ as a Polynomial

By expanding the determinant (2.2.2) with respect to the last column,

$$S_p(n)=\sum_{r=0}^{p}a_{p+1-r}n^{p+1-r}, \tag{2.3.1}$$

with $a_{p+1}=\frac{1}{p+1}$ and, for $r=1,2,\ldots,p$,

$$a_{p+1-r}=\frac{(-1)^r p!}{(p+1-r)!}\begin{vmatrix} -\frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 \\ \frac{1}{3!} & -\frac{1}{2!} & \frac{1}{1!} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{(-1)^{r+1}}{r!} & \frac{(-1)^r}{(r-1)!} & \frac{(-1)^{r-1}}{(r-2)!} & \cdots & \frac{1}{1!} \\ \frac{(-1)^{r+2}}{(r+1)!} & \frac{(-1)^{r+1}}{r!} & \frac{(-1)^r}{(r-1)!} & \cdots & -\frac{1}{2!} \end{vmatrix}$$

$$=\frac{p!}{(p+1-r)!}\begin{vmatrix} \frac{1}{2!} & -\frac{1}{1!} & 0 & \cdots & 0 \\ -\frac{1}{3!} & \frac{1}{2!} & -\frac{1}{1!} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{(-1)^{r+2}}{r!} & \frac{(-1)^{r+1}}{(r-1)!} & \frac{(-1)^r}{(r-2)!} & \cdots & -\frac{1}{1!} \\ \frac{(-1)^{r+3}}{(r+1)!} & \frac{(-1)^{r+2}}{r!} & \frac{(-1)^{r+1}}{(r-1)!} & \cdots & \frac{1}{2!} \end{vmatrix}. \tag{2.3.2}$$

Now multiply the 2^{nd}, 4^{th}, ... columns of the determinant in (2.3.2) by -1 and then multiply the 2^{nd}, 4^{th}, ... rows of the resulting determinant by -1. Then

$$a_{p+1-r}=\frac{p!}{(p+1-r)!}\begin{vmatrix} \frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \cdots & \frac{1}{1!} \\ \frac{1}{(r+1)!} & \frac{1}{r!} & \frac{1}{(r-1)!} & \cdots & \frac{1}{2!} \end{vmatrix}. \tag{2.3.3}$$

Next, recall (see, e.g., [4], p. 323) that the Bernoulli numbers B_j, $j=1,2,\ldots$, can be represented by the determinants

$$B_j=(-1)^j j!\begin{vmatrix} \frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{j!} & \frac{1}{(j-1)!} & \frac{1}{(j-2)!} & \cdots & \frac{1}{1!} \\ \frac{1}{(j+1)!} & \frac{1}{j!} & \frac{1}{(j-1)!} & \cdots & \frac{1}{2!} \end{vmatrix}. \tag{2.3.4}$$

Hence, by (2.3.3) and (2.3.4),

$$a_{p+1-r}=\frac{1}{p+1}(-1)^r\binom{p+1}{r}B_r,\ r=1,2,\ldots,p.$$

Thus, by (2.3.1),

$$S_p(n)=\frac{1}{p+1}\sum_{r=0}^{p}(-1)^r\binom{p+1}{r}B_r n^{p+1-r}. \tag{2.3.5}$$

Since $B_{2r+1}=0\ (r\in N)$,

$$S_p(n)=\frac{1}{p+1}n^{p+1}+\frac{1}{2}n^p+\frac{1}{2}\binom{p}{1}B_2n^{p-1}+\frac{1}{4}\binom{p}{3}B_4n^{p-3}+\cdots,$$

with the last term either containing n or n^2. This is the form in which $S_p(n)$ was given by Jacques Bernoulli in [1].

3. FORMULAS IN TERMS OF POWERS OF $n+1$

3.1 A Recurrence Formula

Let $n\in N$. For $k\in N$, let $S_k(n)=1+2^k+3^k+\cdots+n^k=\Sigma_{r=0}^{n}r^k$ and take $S_0(n)=n+1$. Then, arguing as in the steps leading to (2.1.1) and (2.1.2),

$$\begin{aligned}(n+1)^k&=S_k(n+1)-S_k(n)\\&=\sum_{i=0}^{k}\binom{k}{i}S_i(n)-S_k(n) &(3.1.1)\\&=\sum_{i=0}^{k-1}\binom{k}{i}S_i(n). &(3.1.2)\end{aligned}$$

The equation

$$(S+1)^k-S^k=(n+1)^k, \tag{3.1.3}$$

in which the binomial power is expanded and $S^i\ (i=0,1,2,\ldots,k)$ are then replaced by $S_i(n)$, provides a mnemonic for (3.1.1). Note in particular that (3.1.3) can be obtained from (2.1.3) by merely increasing the values of S, $S-1$, and n by one.

For example, let $k=2$ in (3.1.3). Then $1+2S=(n+1)^2$ and thus $n+1+2S_1(n)=(n+1)^2$, again yielding (2.1.4).

3.2 $S_p(n)$ as a Determinant

Let $p\in N$ and let $k=1,2,\ldots,p,p+1$ in (3.1.2). It follows as in section 2.2, with $(-1)^r$ replaced by 1, that

$$S_p(n)=\frac{1}{(p+1)!}\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & n+1 \\ 1 & \binom{2}{1} & 0 & 0 & \cdots & 0 & (n+1)^2 \\ 1 & \binom{3}{1} & \binom{3}{2} & 0 & \cdots & 0 & (n+1)^3 \\ \vdots & \vdots & \vdots & & & & \vdots \\ 1 & \binom{p}{1} & \binom{p}{2} & & \cdots & \binom{p}{p-1} & (n+1)^p \\ 1 & \binom{p+1}{1} & \binom{p+1}{2} & & \cdots & \binom{p+1}{p-1} & (n+1)^{p+1} \end{vmatrix} \tag{3.2.1}$$

or, alternatively,

$$S_p(n)=p!\begin{vmatrix} \frac{1}{1!} & 0 & 0 & 0 & \cdots & 0 & \frac{n+1}{1!} \\ \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & \cdots & 0 & \frac{(n+1)^2}{2!} \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 & \frac{(n+1)^3}{3!} \\ \vdots & \vdots & \vdots & & & & \vdots \\ \frac{1}{p!} & \frac{1}{(p-1)!} & \frac{1}{(p-2)!} & & \cdots & \frac{1}{1!} & \frac{(n+1)^p}{p!} \\ \frac{1}{(p+1)!} & \frac{1}{p!} & \frac{1}{(p-1)!} & & \cdots & \frac{1}{2!} & \frac{(n+1)^{p+1}}{(p+1)!} \end{vmatrix}. \tag{3.2.2}$$

Note in particular that the determinants in (3.2.1) and (3.2.2) can be obtained from their counterparts in (2.2.1) and (2.2.2) by merely replacing n by $n+1$ in the last column, and replacing all negative entries by their absolute values.

3.3 $S_p(n)$ as a Polynomial

Proceeding as in section 2.3, (3.2.2) can now be employed to establish the formula

$$S_p(n)=\sum_{r=0}^{p} c_{p+1-r}(n+1)^{p+1-r}, \tag{3.3.1}$$

with $c_{p+1}=\frac{1}{p+1}$ and, for $r=1,2,\ldots,p$,

$$c_{p+1-r}=\frac{(-1)^r p!}{(p+1-r)!}\begin{vmatrix} \frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \cdots & \frac{1}{1!} \\ \frac{1}{(r+1)!} & \frac{1}{r!} & \frac{1}{(r-1)!} & \cdots & \frac{1}{2!} \end{vmatrix}. \tag{3.3.2}$$

Hence, by (2.3.4) and (3.3.2),

$$c_{p+1-r}=\frac{1}{p+1}\binom{p+1}{r}B_r,\ r=1,2,\ldots,p.$$

Thus, by (3.3.1),

$$S_p(n)=\frac{1}{p+1}\sum_{r=0}^{p}\binom{p+1}{r}B_r(n+1)^{p+1-r}. \tag{3.3.3}$$

This standard form of $S_p(n)$ is usually established with the aid of the generating function $xe^{tx}/(e^x-1)$ of the Bernoulli polynomials.

The question arises if this method could have led to formulas (2.3.5) and (3.3.3) with $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_3=0$, $B_4=-\frac{1}{30}, \ldots$ without prior knowledge of the Bernoulli numbers. This indeed is the case. First note that, by (2.3.1) and (2.3.3), and (3.3.1) and (3.3.2),

$$S_p(n)=\frac{1}{p+1}\sum_{r=0}^{p}(-1)^r\binom{p+1}{r}b_r n^{p+1-r},$$

and

$$S_p(n)=\frac{1}{p+1}\sum_{r=0}^{p}\binom{p+1}{r}b_r(n+1)^{p+1-r},$$

with $b_0=1$ and, for $r=1,2,\ldots,p$,

$$b_r=(-1)^r r!\begin{vmatrix} \frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \cdots & \frac{1}{1!} \\ \frac{1}{(r+1)!} & \frac{1}{r!} & \frac{1}{(r-1)!} & \cdots & \frac{1}{2!} \end{vmatrix}$$

$$=r!\begin{vmatrix} \frac{1}{1!} & 0 & 0 & \cdots & 0 & 1 \\ \frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \cdots & \frac{1}{1!} & 0 \\ \frac{1}{(r+1)!} & \frac{1}{r!} & \frac{1}{(r-1)!} & \cdots & \frac{1}{2!} & 0 \end{vmatrix}.$$

Now observe that the last determinant differs from that of $S_r(n)$, as obtained by setting $p=r$ in (3.2.2), only with respect to the last column—the entries of b_r, from top to bottom, are 1, 0, ..., 0 while those of $S_r(n)$ are $\frac{n+1}{1!}, \frac{(n+1)^2}{2!}, \ldots, \frac{(n+1)^{r+1}}{(r+1)!}$. It follows [cf. (3.1.2)] that $b_0, b_1, b_2, \ldots$ satisfy the recurrence formula

$$b_0=1,\quad \sum_{i=0}^{r-1}\binom{r}{i}b_i=0\quad (r=2,3,4,\ldots),$$

which generates the numbers $1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, \ldots$, i.e., the Bernoulli numbers.

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