ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-488 Proposed by Paul S. Bruckman, Highwood, IL

The Fibonacci pseudoprimes (or FPP’s) are those composite integers \( n \) with \( \gcd(n, 10) = 1 \) and satisfying the following congruence:

\[
F_{n-e_n} \equiv 0 \pmod{n},
\]

where

\[
e_n = \begin{cases} 
1 & \text{if } n \equiv \pm 1 \pmod{10}, \\
-1 & \text{if } n \equiv \pm 3 \pmod{10}.
\end{cases}
\]

Thus, \( e_n \equiv (\frac{5}{n}) \), a Jacobi symbol.

Given a prime \( p > 5 \), prove that \( u = \frac{1}{3} L_{2p} \) is a FPP if \( u \) is composite.

The Lucas pseudoprimes (or LPP’s) are those composite positive integers \( n \) satisfying the following congruence:

\[
L_n \equiv 1 \pmod{n}.
\]

Given a prime \( p > 5 \), prove that \( u = \frac{1}{3} L_{2p} \) is a LPP if \( u \) is composite.

H-489 Proposed by H.-J. Seiffert, Berlin, Germany

Define the sequences of Pell numbers and Pell-Lucas numbers by

\[
P_0 = 0, \quad P_1 = 1, \quad P_{k+2} = 2P_{k+1} + P_k,
\]

\[
Q_0 = 2, \quad Q_1 = 2, \quad Q_{k+2} = 2Q_{k+1} + Q_k.
\]

Show that

\[
(a) \quad \sum_{n=1}^{\infty} \frac{F_{2^n}P_n}{8(L_{2^n}P_n)^2 - 5(F_{2^n}Q_n)^2} = \frac{1}{12},
\]

\[
(b) \quad \sum_{n=1}^{\infty} \frac{L_{2^n}P_n}{8(L_{2^n}P_n)^2 - 5(F_{2^n}Q_n)^2} = \frac{8 - 3\sqrt{2}}{48}.
\]
ADVANCED PROBLEMS AND SOLUTIONS

SOLUTIONS

A Soft Matrix


Let us define the sequence \( \{ U_n \} \) by

\[
U_0 = 0, \quad U_1 = 1, \quad U_n = PU_{n-1} - QU_{n-2}, \quad n \in \mathbb{Z},
\]

where \( P \) and \( Q \) are nonzero integers. Assuming that \( U_k \neq 0 \), the matrix \( M_k \) is defined by

\[
M_k = \frac{1}{U_k} \begin{pmatrix} U_{k+1} & iQ^{k/2} \\ iQ^{k/2} & -Q^k U_{k-1} \end{pmatrix}, \quad k \geq 1,
\]

where \( i = \sqrt{-1} \).

Express in a closed form the matrix \( M_k^n \), for \( n \geq 0 \).


Solution by H.-J. Seiffert, Berlin, Germany

First, we prove that for all integers \( m, h, \) and \( j \),

\[
U_{m+h} U_{m+j} - U_m U_{m+h+j} = Q^n U_h U_j.
\]  

(1)

We consider the Fibonacci polynomials defined by

\[
F_0(x) = 0, \quad F_1(x) = 1, \quad F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad n \in \mathbb{Z}.
\]

It is easily seen that

\[
U_n = (-Q)^{(n-1)/2} F_n(x), \quad n \in \mathbb{Z},
\]  

(2)

where \( x = P / \sqrt{-Q} \). Multiplying the well-known equation [see A. F. Horadam & Bro. J. M. Mahon, "Pell and Pell-Lucas Polynomials," The Fibonacci Quarterly 23.1 (1985): 12, formula (3.32), where the polynomials \( P_2(x) = F_2(2x) \) are considered]

\[
F_{m+h}(x)F_{m+j}(x) - F_m(x)F_{m+h+j}(x) = (-1)^m F_n(x)F_j(x)
\]

by \((-Q)^{m-1}(b+j)/2 \) and regarding (2), we obtain (1). From (2), it also follows that

\[
U_{-n} = -Q^n U_n, \quad n \in \mathbb{Z}.
\]  

(3)

For \( m = k, \ h = 1, \) and \( j = n \), (1) yields

\[
U_{k+1} U_{k+n} - Q^k U_n = U_k U_{k+n+1}.
\]  

(4)

Similarly, with \( m = k, \ h = n-k, \) and \( j = 1 \),

\[
U_n U_{k+1} - Q^k U_{n-k} = U_k U_{n+1},
\]  

(5)

with \( m = k, \ h = n, \) and \( j = 1-k \),

\[
U_{k+1} - Q^k U_{n-1-k} = U_k U_{n+1},
\]  

(6)
and finally, with $m = n$, $h = 1 - k$, and $j = k - n$, (1) gives
\[
U_n + Q^n U_{n-1-k} U_{k-n} = U_k U_{n+1-k}
\]
or, by (3),
\[
U_n - Q^n U_{n-1-k} U_{k-n} = U_k U_{n+1-k}.
\]
With the help of (4)-(7), it is easily proved by induction on $n$ that
\[
M_k^n = \frac{1}{U_k} \left( \frac{U_{k+n}}{U_n} iQ^{k/2} U_n - iQ^{k/2} U_{n-k} \right), \quad n \geq 1.
\]

Using (3), it is easily seen that this equation also holds for $n = 0$.

*Also solved by P. Bruckman, A. G. Shannon, and the proposer.*

Get It off Your Chess


Professional chess players today use the algebraic chess notation. This is based upon the algebraic numbering of the chessboard. The eight letters $a$ through $h$ and the eight digits 1 through 8 are used to form sixty-four combinations of a letter and a digit which are called "symbol pairs." Those sixty-four symbol pairs are used to represent the sixty-four squares of the chessboard.

Develop a viable arithmetic numbering of the chessboard, as follows:

(a) Use twenty-five letters of the alphabet (all except $U$) and nine decimal digits (all except zero) to form 225 symbol pairs; choose sixty-four of those symbol pairs to represent the sixty-four squares of the chessboard.

(b) There are thirty-six squares from which a King can move to eight other squares. Let the nine symbol pairs representing the location of the King and the squares to which it can move contain all nine decimal digits.

(c) There are sixteen squares from which a Knight can move to eight other squares. A Queen located on one of those sixteen squares, moving one or two squares, can go to sixteen other squares. Let the twenty-five symbol pairs representing the location of the Knight or the Queen and the squares to which the Knight or the Queen can move contain all twenty-five letters of the alphabet.

(d) Let the algebraic Square $a8$ (the original location of Black’s Queen Rook) correspond to the arithmetic Square $A1$; let the algebraic Square $h1$ (the original location of White’s King Rook) correspond to the arithmetic Square $Z9$.

**Solution by Leonard A. G. Dresel, Reading, England**

Consider the basic $3 \times 3$ and $5 \times 5$ patterns given by:

<table>
<thead>
<tr>
<th>1 2 3</th>
<th>A B C D E</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 5 6</td>
<td>F G H I J</td>
</tr>
<tr>
<td>7 8 9</td>
<td>K L M* N O</td>
</tr>
<tr>
<td>5 6 7</td>
<td>P Q R S T</td>
</tr>
</tbody>
</table>

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Repeating these patterns across the 8 × 8 board, left to right and then top to bottom, and superposing them, we can satisfy conditions (b) and (c). To satisfy (d) and obtain Z9 in the bottom right corner, we exchange 5 with 9 and M with Z in the basic patterns. Thus, we arrive at a viable numbering given by:

\[
\begin{array}{cccccccc}
F4 & G9 & H6 & I4 & J9 & F6 & G4 & H9 \\
K7 & L8 & Z5 & N7 & O8 & K5 & L7 & Z8 \\
P1 & Q2 & R3 & S1 & T2 & P3 & Q1 & R2 \\
V4 & W9 & X6 & Y4 & M9 & V6 & W4 & X9 \\
A7 & B8 & C5 & D7 & E8 & A5 & B7 & C8 \\
F1 & G2 & H3 & I1 & J2 & F3 & G1 & H2 \\
K4 & L9 & Z6 & N4 & O9 & K6 & L4 & Z9 \\
\end{array}
\]

Since 3 and 5 are co-prime, the repeating patterns ensure that no alpha-numeric combination occurs more than once.

The solution is not unique, as we can choose the modified basic patterns in \((7! \times 23!\) ways to satisfy condition (d).

Also solved by P. Bruckman, J. Hendel, and the proposer.

**Pell-Mell**

**H-476** Proposed by H.-J. Seiffert, Berlin, Germany  
(Vol. 31, no. 2, May 1993)

Define the Pell numbers by \(P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}, \) for \(n \geq 2\). Show that, for all positive integers \(n\),

\[
P_n = \sum_{k=0}^{n-1} (-1)^{(3k+3-2n)/4} 2^{[3k/2]} \binom{n+k}{2k+1},
\]

where \([ \ ]\) denotes the greatest integer function.

Solution by Paul S. Bruckman, Highwood, IL

Let \(S_n\) denote the sum given in the statement of the problem. It is easily shown that

\[
\frac{x}{f(x)} = \sum_{n=1}^{\infty} P_n x^n, \ |x| < \sqrt{2} - 1,
\]

where

\[
f(x) = 1 - 2x - x^2.
\]

To prove that \(S_n = P_n, n = 1, 2, \ldots\), it will suffice to show that \(g(x) = \frac{x}{f(x)}\), where

\[
g(x) = \sum_{n=1}^{\infty} S_n x^n;
\]

presumably, this is to be valid for all \(x\) with \(|x| < \sqrt{2} - 1\).

As usual with generating function techniques, we will ignore questions of convergence (which should be considered, a posteriori). Then

\[
g(x) = \sum_{k, m \geq 0} x^{m+k+1} (-1)^{(3k+3-2m)/4} 2^{[3k/2]} \binom{m+2k+1}{2k+1}.
\]
Letting \( m = 2u \) or \( m = 2u + 1 \), we obtain

\[
g(x) = \sum_{u \geq 0} x^{2u+k+1} (-1)^{(k+1)/4} x^{u+2k+1} \left( \frac{2u + 2k + 1}{2u} \right)
+ \sum_{u \geq 0} x^{2u+k+2} (-1)^{(k-1)/4} x^{u+2k+2} \left( \frac{2u + 1 + 2k + 1}{2u+1} \right)
\]

\[
= \sum_{j \geq 1} \frac{x^j}{4} (-1)^{[j/4]} 2^{[j/2]} \sum_{u \geq 0} (-1)^u x^{2u} \left( \frac{-2j}{2u} \right)
- \sum_{j \geq 1} \frac{x^j}{4} (-1)^{[j/2]} 2^{[j/2]} \sum_{u \geq 0} (-1)^u x^{2u+1} \left( \frac{-2j}{2u+1} \right)
\]

Now

\[
\sum_{u \geq 0} (-1)^u x^{2u} \left( \frac{-2j}{2u} \right) = \sum_{v \geq 0} (ix)^v e_v \left( \frac{-2j}{v} \right),
\]

where \( e_v = \frac{1}{2} (1 + (-1)^v) \), which equals \( \frac{1}{2} (\theta^{2j} + \overline{\theta}^{2j}) = \text{Re}(\theta^{2j}) \), with \( \theta = 1 + ix \). Likewise,

\[
\sum_{u \geq 0} (-1)^u x^{2u+1} \left( \frac{-2j}{2u+1} \right) = -j \sum_{v \geq 0} (ix)^v \alpha_v \left( \frac{-2j}{v} \right),
\]

where \( \alpha_v = \frac{1}{2i} (1 - (-1)^v) \), which equals \( \frac{1}{2i} (\theta^{2j} - \overline{\theta}^{2j}) = \text{Im}(\theta^{2j}) \). Therefore,

\[
g(x) = \text{Re}(U(x) + iV(x)), \quad (4)
\]

where

\[
U(x) = \sum_{j \geq 1} \frac{x^j}{4} (-1)^{[j/2]} 2^{[j/2]} \theta^{2j}, \quad (5)
\]

\[
V(x) = \sum_{j \geq 1} \frac{x^j}{4} (-1)^{[j/2]} 2^{[j/2]} \theta^{-2j}. \quad (6)
\]

Making the substitutions \( j = 4i + r \), where \( i \geq 0 \) and \( r = 1, 2, \) or \( 4 \) in (5), \( r = 2, 3, 4 \) in (6), we find that

\[
U(x) = (x / \theta^2 + 2x^2 / \theta^4 - 16x^4 / \theta^6) \cdot h(x), \quad (7)
\]

\[
V(x) = (2x^2 / \theta^4 + 8x^3 / \theta^6 + 16x^4 / \theta^8) \cdot h(x), \quad (8)
\]

where

\[
h(x) = \sum_{i=0}^{\infty} (-1)^i 2^i x^i \theta^{-8i}. \quad (9)
\]

Thus,

\[
h(x) = (1 + 64x^4 / \theta^8)^{-1} = \frac{\theta^8}{\theta^8 + 64x^4},
\]

from which we obtain

\[
U(x) = \frac{x}{\theta^8 + 64x^4} (\theta^6 + 2x^4 - 16x^3), \quad (10)
\]

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\[ V(x) = \frac{2x^2}{\theta^8 + 64x^4} \cdot (\theta^4 + 4x\theta^2 + 8x^2). \] (11)

As we may verify, \( \theta^8 + 64x^4 = (\theta^4 + 4x\theta^2 + 8x^2)(\theta^4 - 4x\theta^2 + 8x^2) \) and \( \theta^6 + 2x\theta^4 - 16x^3 = (\theta^4 + 4x\theta^2 + 8x^2)(\theta^2 - 2x) \). Thus,
\[
U(x) = \frac{x(\theta^2 - 2x)}{\theta^4 - 4x\theta^2 + 8x^2}, \quad V(x) = \frac{2x^2}{\theta^4 - 4x\theta^2 + 8x^2}.
\] (12)

Next, we observe that \( \theta^2 = 1+2i - x^2 = 1 - 2x - x^2 + 2x(1+i) = f(x) + 2x(1+i) \). Also, we have \( \theta^4 = (f(x))^2 + 4xf(x)(1+i) + 4x^2 \cdot 2i = f(x)[f(x) + 4ix] + 4xf(x) + 8ix^2 \), from which it follows that \( \theta^4 - 4x\theta^2 + 8x^2 = f(x)[f(x) + 4ix] \). Then
\[
U(x) + iV(x) = \frac{x(\theta^2 - 2x + 2ix)}{f(x)[f(x) + 4ix]} = \frac{x[f(x) + 4ix]}{f(x)[f(x) + 4ix]} = \frac{x}{f(x)}.
\]

Hence, we see that \( U(x) + iV(x) \) is real, so that
\[ \text{Re}(U(x) + iV(x)) = U(x) + iV(x) = g(x) = \frac{x}{f(x)} \]. Q.E.D.

\[
\bullet\bullet\bullet
\]